

Tensor product for symmetric monoidal categories

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Abstract

We introduce a tensor product for symmetric monoidal categories with the following properties. Let SMC denote the 2-category with objects small symmetric monoidal categories, arrows *symmetric monoidal functors* and 2-cells monoidal natural transformations. Our tensor product together with a suitable unit is part of a structure on SMC that is a 2-categorical version of the symmetric monoidal closed categories. This structure is surprisingly simple. In particular the arrows involved in the associativity and symmetry laws for the tensor and in the unit cancellation laws are 2-natural and satisfy coherence axioms which are *strictly commuting diagrams*. We also show that the category quotient of SMC by the congruence generated by its 2-cells admits a symmetric monoidal closed structure.

1 Summary of results

Thomason's famous result claims that symmetric monoidal categories model all connective spectra [Tho95]. The discovery of a symmetric monoidal structure on the category of structured spectra [EKMM97] suggests that a similar structure should exist on an adequate category with symmetric monoidal categories as objects. The first aim of this work is to give a reasonable candidate for a tensor product of symmetric monoidal categories.

We define such a tensor product for two symmetric monoidal categories by means of a generating graph and relations. It has the following properties. Let SMC denote the 2-category with objects symmetric monoidal categories, with 1-cells *symmetric monoidal functors* and 2-cells monoidal natural transformations. The tensor product yields a 2-functor $SMC \times SMC \rightarrow SMC$. This one is part of a 2-categorical structure on SMC that is 2-categorical version of the symmetric monoidal closed categories. Moreover this structure is rather simple since:

- its "canonical" arrows, i.e those involved for the associativity, the symmetry and the left and right unit cancellation laws, are *2-natural*;
- all coherence axioms for the above arrows are *strictly commuting diagrams*.

Actually this last point was quite unexpected. Eventually from the above structure one can deduce a symmetric monoidal closed structure on the category $SMC/_\sim$ quotient of SMC by the congruence \sim generated by its 2-cells.

Here are now, in brief, the technical results in the order in which they occur in the paper. The existence of an *internal hom*, a *tensor* and a *unit* for SMC and the fundamental properties defining and relating those are established first. From this, further properties of the above structure can be derived, such as the existence of associativity, symmetry and unit laws involving 2-natural arrows satisfying coherence axioms. The proofs are rather computational for establishing a few key facts

but become hopefully convincingly short and abstract for the rest of the paper.

Sections 2,3 and 4 are preliminaries. A first point is that the 2-category SMC admits internal homs in the following sense. For any symmetric monoidal categories \mathcal{A} and \mathcal{B} , the mere category $SMC(\mathcal{A}, \mathcal{B})$ admits a symmetric monoidal structure denoted $[\mathcal{A}, \mathcal{B}]$. This data extends to a 2-functor $\underline{Hom} : SMC^{op} \times SMC \rightarrow SMC$ (Proposition 9.7). Moreover the classical isomorphism $\mathbf{Cat}(\mathcal{A}, \mathbf{Cat}(\mathcal{B}, \mathcal{C})) \cong \mathbf{Cat}(\mathcal{B}, \mathbf{Cat}(\mathcal{A}, \mathcal{C}))$ induces a 2-natural isomorphism $D : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \cong [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$ between 2-functors $SMC^{op} \times SMC^{op} \times SMC \rightarrow SMC$ (Proposition 10.4). The isomorphism D is natural even in a stronger sense as it is precisely stated in Lemma 10.7 and 10.8. The results above are presented in the sections from 5 to 10. A brief section 11 treats the functors $ev_a : [\mathcal{A}, \mathcal{B}] \rightarrow \mathcal{B}$ given by the evaluation at some object a of \mathcal{A} . They are the images of objects a by the functor $\mathcal{A} \rightarrow [[\mathcal{A}, \mathcal{B}], \mathcal{B}]$ that corresponds via D to the identity at $[\mathcal{A}, \mathcal{B}]$.

The tensor $\mathcal{A} \otimes \mathcal{B}$ of any symmetric monoidal categories \mathcal{A} and \mathcal{B} is defined in section 12. In section 14, two functors are defined for any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} , namely

$$En : SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$$

and

$$Rn : SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \rightarrow SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]).$$

They both admit symmetric monoidal structures yielding a monoidal adjunction

$$En \dashv Rn : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

with $Rn \circ En = 1$. It is also shown that the functors $En_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ factor as

$$SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \xrightarrow{\cong} StrSMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \longrightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$$

where $StrSMC$ denotes the sub-2-category of SMC with the same objects but with *strict* functors as 1-cells and monoidal natural transformations as 2-cells (Proposition 14.2 and Corollary 14.5). As explained in section 15, this universal characterisation of the tensor of symmetric monoidal categories serves to define its extension to a 2-functor $SMC \times SMC \rightarrow SMC$. Then the naturality issues for the collection of arrows Rn and En are treated in section 16.

Technical lemmas regarding the interaction between the internal hom, the tensor and the isomorphism D in SMC are grouped in section 17.

The free symmetric monoidal category over the one point category, denoted \mathcal{I} , is studied in section 18. By the universal property defining \mathcal{I} , for any symmetric monoidal category \mathcal{A} , there is a unique strict symmetric monoidal functor $v : \mathcal{I} \rightarrow [\mathcal{A}, \mathcal{A}]$ sending the generator of \mathcal{I} to the identity $1 : \mathcal{A} \rightarrow \mathcal{A}$. The symmetric monoidal functor $v^* : \mathcal{A} \rightarrow [\mathcal{I}, \mathcal{A}]$ corresponding to v via D will reveal useful and, as a 1-cell in SMC , admits as right adjoint in SMC the functor $ev_\star : [\mathcal{I}, \mathcal{A}] \rightarrow \mathcal{A}$ evaluation at the generator \star of \mathcal{I} .

Then what can be seen as a “lax symmetric monoidal 2-categorical structure” is defined on SMC . The tensor and unit are known.

Arrows for the associativity, unit and symmetry laws are introduced in section 19: A' for associativity, L' , R' for the left and right unit cancellations and S' for the symmetry. It is shown that the usual coherence axioms for symmetric monoidal categories hold for SMC (see points 19.9, 19.14, 19.2, 19.11 and 19.1) and that the canonical arrows are 2-natural with “lax” inverses (points 19.7, 19.8, 19.17).

Eventually last section 20 recaps everything for proving the symmetric monoidal closeness of the category SMC/\sim . This last point is Theorem 20.2.

The following remark is in order. A related result was found by M.Hyland and J.Power in their paper [HyPo02] which treats in particular the 2-category with objects symmetric monoidal categories, *but with strong functors as 1-cells*, and monoidal natural transformations as 2-cells. This was done though in a general 2-categorical setting extending A.Kock's work on commutative monads. Hyland and Power discovered a 2-categorical structure, namely that of *pseudo-monoidal closed 2-category*, which exists on the 2-category of T -algebras with their pseudo morphisms for any *pseudo-commutative doctrine* T on **Cat**. Their pseudo monoidal closed structure is also a 2-categorical generalisation of the Eilenberg-Kelly's monoidal closed structure [EiKe66], with a tensor which is a *pseudo-functor*, with canonical morphisms which are *pseudo-natural equivalences* and coherence diagrams which commute *up to coherent isomorphic 2-cells*. In particular the tensor for T -algebras in [HyPo02] is defined from a collection of left biadjoints $\mathcal{A} \otimes -$ to internal homs $[\mathcal{A}, -]$. Actually this point does not seem to generalise to the 2-categories of T -algebras with their lax morphisms. Eventually the tensors here and in [HyPo02] are not isomorphic as symmetric monoidal categories due to the fact that we consider lax morphisms of algebras whereas Hyland and Power considered the pseudo ones.

The author's view is that the case of symmetric monoidal categories together with monoidal functors should be elucidated before considering any generalisation in the line of Kock's or Hyland-Power's works.

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2 Preliminaries

The purpose of this section is to recall basic notions, introduce notations and give references to some results used through the paper.

We shall write **Set** for the category of sets, and **Cat** for the 2-category of *small* categories.

- A *monoidal category* $(\mathcal{A}, \otimes, I, ass, r, l)$ consists of a category \mathcal{A} together with:
- a functor $\otimes : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$, sometimes denoted for convenience by *Ten*;
 - an object I of \mathcal{A} , sometimes explicitly written $I_{\mathcal{A}}$;
 - natural isomorphisms $ass_{a,b,c} : a \otimes (b \otimes c) \rightarrow (a \otimes b) \otimes c$, $r_a : a \otimes I \rightarrow a$ and $l_a : I \otimes a \rightarrow a$ that satisfy the two coherence axioms 2.1 and 2.2 below.

2.1 The diagram in \mathcal{A}

$$\begin{array}{ccccc}
 a \otimes (b \otimes (c \otimes d)) & \xrightarrow{ass} & (a \otimes b) \otimes (c \otimes d) & \xrightarrow{ass} & ((a \otimes b) \otimes c) \otimes d \\
 \downarrow 1 \otimes ass & & & & \uparrow ass \otimes 1 \\
 a \otimes ((b \otimes c) \otimes d) & \xrightarrow{ass} & (a \otimes (b \otimes c)) \otimes d & &
 \end{array}$$

commutes for any a, b, c and d .

2.2 The diagram in \mathcal{A}

$$\begin{array}{ccc}
 a \otimes (I \otimes b) & \xrightarrow{ass} & (a \otimes I) \otimes b \\
 & \searrow 1 \otimes l \quad \swarrow r \otimes 1 & \\
 & a \otimes b &
 \end{array}$$

commutes for any a and b .

A symmetric monoidal category $(\mathcal{A}, \otimes, I, ass, l, r, s)$ consists of a monoidal category $(\mathcal{A}, \otimes, I, ass, r, l)$ with a natural transformation $s_{a,b} : a \otimes b \cong b \otimes a$ such that the following coherence axioms 2.3, 2.4 and 2.5 are satisfied.

2.3 The composite in \mathcal{A}

$$a \otimes b \xrightarrow{s} b \otimes a \xrightarrow{s} a \otimes b$$

is the identity at $a \otimes b$ for any a and b .

2.4 The diagram in \mathcal{A}

$$\begin{array}{ccccc}
 a \otimes (b \otimes c) & \xrightarrow{ass} & (a \otimes b) \otimes c & \xrightarrow{s} & c \otimes (a \otimes b) \\
 1 \otimes s \downarrow & & & & \downarrow ass \\
 a \otimes (c \otimes b) & \xrightarrow{ass} & (a \otimes c) \otimes b & \xrightarrow{s \otimes 1} & (c \otimes a) \otimes b
 \end{array}$$

commutes for any a, b, c .

2.5 The diagram in \mathcal{A}

$$\begin{array}{ccc}
 a \otimes I & \xrightarrow{s} & I \otimes a \\
 & \searrow r \quad \swarrow l & \\
 & a &
 \end{array}$$

commutes for any a .

A symmetric monoidal category is said *strict* when its isomorphisms ass , l , r and s are identities.

For convenience, we adopt the view that all the monoidal categories considered further are by default *small*.

A *monoidal functor* between monoidal categories \mathcal{A} and \mathcal{B} , consists of a triple (F, F^0, F^2) where:

- $F : \mathcal{A} \rightarrow \mathcal{B}$ is a functor;
 - F^0 is an arrow $I \rightarrow F(I)$ in \mathcal{B} ;
 - F^2 is a natural transformation $F^2_{x,y} : Fx \otimes Fy \rightarrow F(x \otimes y)$ in x and y ;
- and that is subject to the axioms 2.6, 2.7, 2.8 below.

2.6 For any objects a, b, c of \mathcal{A} , the diagram in \mathcal{B}

$$\begin{array}{ccc}
 Fa \otimes (Fb \otimes Fc) & \xrightarrow{ass_{Fa, Fb, Fc}} & (Fa \otimes Fb) \otimes Fc \\
 1 \otimes F^2_{b,c} \downarrow & & \downarrow F^2_{a,b} \otimes 1 \\
 Fa \otimes F(b \otimes c) & & F(a \otimes b) \otimes Fc \\
 F^2_{a,b \otimes c} \downarrow & & \downarrow F^2_{a \otimes b, c} \\
 F(a \otimes (b \otimes c)) & \xrightarrow{F(ass_{a,b,c})} & F((a \otimes b) \otimes c)
 \end{array}$$

commutes.

2.7 For any object a in \mathcal{A} , the diagram in \mathcal{B}

$$\begin{array}{ccc} Fa \otimes I & \xrightarrow{r_{Fa}} & Fa \\ 1 \otimes F^0 \downarrow & & \uparrow Fr_a \\ Fa \otimes FI & \xrightarrow{F^2_{a,I}} & F(a \otimes I) \end{array}$$

commutes.

2.8 For any object a in \mathcal{A} , the diagram in \mathcal{B}

$$\begin{array}{ccc} I \otimes Fa & \xrightarrow{l} & Fa \\ F^0 \otimes 1 \downarrow & & \uparrow Fl \\ F(I) \otimes Fa & \xrightarrow{F^2_{I,a}} & F(I \otimes a) \end{array}$$

commutes.

A monoidal functor (F, F^0, F^2) is *strict* when the arrows F^0 and the F^2 are identities and it is *strong* when those are isomorphisms. When the monoidal categories \mathcal{A} and \mathcal{B} are symmetric, a monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$ is *symmetric* when it satisfies the following axiom.

2.9 For any objects a, b of \mathcal{A} , the diagram in \mathcal{B}

$$\begin{array}{ccc} Fa \otimes Fb & \xrightarrow{F^2_{a,b}} & F(a \otimes b) \\ s_{Fa, Fb} \downarrow & & \downarrow F s_{a \otimes b} \\ Fb \otimes Fa & \xrightarrow{F^2_{b,a}} & F(b \otimes a) \end{array}$$

commutes.

Though symmetric monoidal functors are triples we will sometimes just mention “the symmetric monoidal F ” to mean actually the triple (F, F^0, F^2) .

Given monoidal functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$, a natural transformation $\sigma : F \rightarrow G$ is *monoidal* when it satisfies axioms 2.10 and 2.11 below.

2.10 For any objects a, b of \mathcal{A} , the diagram in \mathcal{B}

$$\begin{array}{ccc} Fa \otimes Fb & \xrightarrow{F^2_{a,b}} & F(a \otimes b) \\ \sigma_a \otimes \sigma_b \downarrow & & \downarrow \sigma_{a \otimes b} \\ Ga \otimes Gb & \xrightarrow{G^2_{a,b}} & G(a \otimes b) \end{array}$$

commutes.

2.11 The diagram in \mathcal{B}

$$\begin{array}{ccc} I & \xrightarrow{F^0} & F(I) \\ & \searrow G^0 & \downarrow \sigma_I \\ & & G(I) \end{array}$$

commutes.

Recall that symmetric monoidal categories, symmetric monoidal functors and monoidal transformations form a 2-category that we denote SMC . In particular given two symmetric monoidal functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$, the symmetric monoidal structure of the composite GF is the following:

- the arrow $(GF)^0$ is $I_{\mathcal{C}} \xrightarrow{G^0} G(I_{\mathcal{B}}) \xrightarrow{G(F^0)} GF(I_{\mathcal{A}})$;
- for any objects a, a' in \mathcal{A} , the arrow $(GF)_{a,a'}^2$ is

$$GF(a) \otimes GF(a') \xrightarrow{G_{Fa, Fa'}^2} G(F(a) \otimes F(a')) \xrightarrow{G(F_{a,a'}^2)} GF(a \otimes a').$$

We write $StrSMC$ for the sub-2-category of SMC , with the same objects but with 1-cells the *strict* symmetric monoidal functors and monoidal transformations as 2-cells.

One has a forgetful 2-functor $U : SMC \rightarrow \mathbf{Cat}$ sending symmetric monoidal categories, symmetric monoidal functors and monoidal transformations respectively to their underlying categories, functors and natural transformations. There exists a notion of free symmetric monoidal category in the following sense: the forgetful 2-functor $StrSMC \rightarrow \mathbf{Cat}$ admits a left 2-adjoint.

Remember from enriched category theory the following ([Kel82] p.43).

Lemma 2.12 *For any symmetric monoidal closed complete and cocomplete category \mathcal{V} and any \mathcal{V} -functor $F : \mathcal{A} \rightarrow \mathcal{B}$ the collection of its components $F_{a,a'} : \mathcal{A}(a, a') \rightarrow \mathcal{B}(Fa, Fa')$ in \mathcal{V} is \mathcal{V} -natural in a and a' .*

As a consequence the collection of functors $U_{\mathcal{A}, \mathcal{B}} : SMC(\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{Cat}(U\mathcal{A}, U\mathcal{B})$, components in \mathcal{A} and \mathcal{B} of the above forgetful 2-functor U , is 2-natural in \mathcal{A} and \mathcal{B} .

There is an important result regarding coherence for symmetrical monoidal categories, which is explained for instance in the second version of Mac Lane's handbook [McLaCWM] in chapter 11. We will use implicitly this result, sometimes mentioning the coherence for symmetric monoidal categories. Also we will often omit to explicit the canonical isomorphisms in symmetric monoidal categories when they are the expected ones.

G.M. Kelly characterised in [Kel74] the *symmetric monoidal adjunctions* which are the adjunctions in the 2-category SMC . In particular let us recall that *a symmetric monoidal functor is left adjoint in SMC if and only if it is strong and left adjoint in \mathbf{Cat}* . Though monoidal adjunctions are a central concept in the present paper, we will use only the following result from Kelly.

Proposition 2.13 *Given any symmetric monoidal functor $(G, G^0, G^2) : \mathcal{B} \rightarrow \mathcal{A}$ that admits a left adjoint F in \mathbf{Cat} , G admits a left adjoint in the 2-category SMC if and only if the following two conditions hold:*

- (1) *For any objects a, a' in \mathcal{A} , the arrows in \mathcal{B}*

$$F(a \otimes a') \xrightarrow{F(\eta_a \otimes \eta_{a'})} F(GFa \otimes GFa') \xrightarrow{F(G_{Fa, Fa'}^2)} FG(Fa \otimes Fa') \xrightarrow{\epsilon_{Fa \otimes Fa'}} Fa \otimes Fa'$$

are invertible;

- (2) *The arrow in \mathcal{B}*

$$F(I) \xrightarrow{F(G^0)} FG(I) \xrightarrow{\epsilon_I} I$$

is invertible.

In this case, the functor F admits a strong monoidal structure with the $F_{a,a'}^2$ and F^0 respectively inverses of the above arrows (1) and (2), and (F, F^0, F^2) is the left adjoint of G in SMC .

We will use the Yoneda machinery in the enriched context. Let us consider a base symmetric monoidal closed category \mathcal{V} that is complete and cocomplete. (The base considered further in the paper is always the cartesian closed **Cat**.) Recall that for any \mathcal{V} -functor $F : \mathcal{A} \rightarrow \mathcal{V}$ the Yoneda lemma defines a bijection between \mathcal{V} -natural transformations in the argument a , $\mathcal{A}(a, k) \rightarrow Fa$ and arrows $I \rightarrow Fk$ in \mathcal{V} . We will use extensively the two following results from [Kel82].

Lemma 2.14 *Given any \mathcal{V} -functors $F : \mathcal{A}^{op} \otimes \mathcal{B} \rightarrow \mathcal{V}$ and $K : \mathcal{A} \rightarrow \mathcal{B}$, with a collection of arrows $\phi_{a,b} : \mathcal{A}(a, Kb) \rightarrow_a F(a, b)$, \mathcal{V} -natural in the argument a for each object b in \mathcal{B} , this collection is also \mathcal{V} -natural in b if and only if the collection of arrows $I \rightarrow F(Kb, b)$ corresponding by Yoneda is \mathcal{V} -natural in b .*

Lemma 2.15 (Representability with parameters) *For any \mathcal{V} -functor $F : \mathcal{A}^{op} \otimes \mathcal{B} \rightarrow \mathcal{V}$ with a collection of objects $K(b)$ and \mathcal{V} -natural isomorphisms $\phi_b : \mathcal{A}(a, Kb) \cong_a F(a, b)$ in the argument a for each object b of \mathcal{B} , there is a unique extension of K into a \mathcal{V} -functor $\mathcal{B} \rightarrow \mathcal{A}$ such that the collection of isomorphisms ϕ_b is also \mathcal{V} -natural in the argument b .*

3 Terminal object and product in SMC

The terminal category **1** (with one object and one arrow) admits a unique symmetric monoidal structure, which is the strict one. Still denoting by **1** the latter object in SMC , one has an isomorphism in **Cat**, 2-natural in \mathcal{A}

3.1

$$SMC(\mathcal{A}, \mathbf{1}) \cong \mathbf{1}.$$

In particular **1** is also the terminal object of the underlying category of SMC . For any symmetric monoidal category \mathcal{B} , the constant functor $\Delta_{I_{\mathcal{B}}} : \mathbf{1} \rightarrow \mathcal{B}$ to $I_{\mathcal{B}}$ admits a strong symmetric monoidal structure as follows. For the unique object \star of **1**, the arrow $(\Delta_{I_{\mathcal{B}}})_{\star, \star}^2$ is defined as the canonical one $r_I = l_I : I \otimes I \rightarrow I$ in \mathcal{B} and the arrow $(\Delta_{I_{\mathcal{B}}})^0$ is defined as the identity at $I_{\mathcal{B}}$. So that for any symmetric monoidal categories \mathcal{A} and \mathcal{B} the constant functor $\mathcal{A} \rightarrow \mathcal{B}$ to $I_{\mathcal{B}}$ admits a symmetric strong monoidal structure as the composite $\mathcal{A} \longrightarrow \mathbf{1} \xrightarrow{\Delta_{I_{\mathcal{B}}}} \mathcal{B}$ in SMC .

The cartesian product of categories satisfies the following property. For any categories \mathcal{A} , \mathcal{B} and \mathcal{C} , one has an isomorphism of categories

3.2

$$\mathbf{Cat}(\mathcal{A}, \mathcal{B} \times \mathcal{C}) \cong \mathbf{Cat}(\mathcal{A}, \mathcal{B}) \times \mathbf{Cat}(\mathcal{A}, \mathcal{C})$$

and for some arbitrary fixed \mathcal{B} and \mathcal{C} this collection of isomorphisms is 2-natural in \mathcal{A} . There is a unique 2-functor, say the *cartesian product* 2-functor, $- \times - : \mathbf{Cat} \times \mathbf{Cat} \rightarrow \mathbf{Cat}$ that makes the isomorphism 3.2 also 2-natural in \mathcal{B} and \mathcal{C} . (This can be checked in an ad-hoc way but this also results from the Yoneda Lemma in its enriched version by considering 2-categories and 2-functors as categories and functors enriched over the cartesian closed category **Cat**.)

When the categories \mathcal{B} and \mathcal{C} have a symmetric monoidal structure, their cartesian product admits the following “pointwise” symmetric monoidal structure.

- The unit $I_{\mathcal{B} \times \mathcal{C}} = (I_{\mathcal{B}}, I_{\mathcal{C}})$. Equivalently one can define the constant functor $\Delta_{I_{\mathcal{B} \times \mathcal{C}}} : \mathbf{1} \rightarrow \mathcal{B} \times \mathcal{C}$ to $I_{\mathcal{B} \times \mathcal{C}}$ as the composite in **Cat**

$$\mathbf{1} \xrightarrow{\Delta} \mathbf{1} \times \mathbf{1} \xrightarrow{\Delta_{I_{\mathcal{B}}} \times \Delta_{I_{\mathcal{C}}}} \mathcal{B} \times \mathcal{C}$$

where Δ stands for the diagonal functor. Note that the above composite is the unique functor $F : \mathbf{1} \rightarrow \mathcal{B} \times \mathcal{C}$ such that $F \circ p_{\mathcal{B}} = \Delta_{I_{\mathcal{B}}}$ and $F \circ p_{\mathcal{C}} = \Delta_{I_{\mathcal{C}}}$, where $p_{\mathcal{B}}$ and $p_{\mathcal{C}}$ are respectively the

projections $\mathcal{B} \times \mathcal{C} \rightarrow \mathcal{B}$ and $\mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$.

- The tensor $Ten : (\mathcal{B} \times \mathcal{C}) \times (\mathcal{B} \times \mathcal{C}) \rightarrow (\mathcal{B} \times \mathcal{C})$ is the composite functor

$$(\mathcal{B} \times \mathcal{C}) \times (\mathcal{B} \times \mathcal{C}) \xrightarrow{\cong} (\mathcal{B} \times \mathcal{B}) \times (\mathcal{C} \times \mathcal{C}) \xrightarrow{Ten \times Ten} \mathcal{B} \times \mathcal{C}$$

which is also the unique functor $F : (\mathcal{B} \times \mathcal{C}) \times (\mathcal{B} \times \mathcal{C}) \rightarrow \mathcal{B} \times \mathcal{C}$ such that $p_{\mathcal{B}} \circ F = Ten \circ (p_{\mathcal{B}} \times p_{\mathcal{B}})$ and $p_{\mathcal{C}} \circ F = Ten \circ (p_{\mathcal{C}} \times p_{\mathcal{C}})$. This is to say that for any objects b, b' in \mathcal{B} and c, c' in \mathcal{C} , the object $(b, c) \otimes (b', c')$ is $(b \otimes b', c \otimes c')$ and for any arrows $f : b_1 \rightarrow b_2$, $f' : b'_1 \rightarrow b'_2$ in \mathcal{B} and $g : c_1 \rightarrow c_2$, $g' : c'_1 \rightarrow c'_2$ in \mathcal{C} , the arrow $(f, g) \otimes (f', g') : (b_1, c_1) \otimes (b'_1, c'_1) \rightarrow (b_2, c_2) \otimes (b'_2, c'_2)$ is $(f \otimes f', g \otimes g') : (b_1 \otimes b'_1, c_1 \otimes c'_1) \rightarrow (b_2 \otimes b'_2, c_2 \otimes c'_2)$.

- For any objects b, b', b'' in \mathcal{B} and c, c', c'' in \mathcal{C} , the arrow $ass_{(b,c),(b',c'),(b'',c')}$ is

$$\begin{array}{c} (b, c) \otimes ((b', c') \otimes (b'', c'')) \\ \parallel \\ ((b \otimes (b' \otimes b''), c \otimes (c' \otimes c'')) \\ \downarrow (ass_{b,b',b'',c,c',c''}) \\ ((b \otimes b') \otimes b'', (c \otimes c') \otimes c'') \\ \parallel \\ ((b, c) \otimes (b', c')) \otimes (b'', c''). \end{array}$$

- For any objects b in \mathcal{B} and c in \mathcal{C} , the arrow $r_{(b,c)}$ is defined as

$$(b, c) \otimes (I_{\mathcal{B}}, I_{\mathcal{C}}) = (b \otimes I_{\mathcal{B}}, c \otimes I_{\mathcal{C}}) \xrightarrow{(r_{\mathcal{B}}, r_{\mathcal{C}})} (b, c)$$

and $l_{(b,c)}$ is

$$(I_{\mathcal{B}}, I_{\mathcal{C}}) \otimes (b, c) = (I_{\mathcal{B}} \otimes b, I_{\mathcal{C}} \otimes c) \xrightarrow{(l_{\mathcal{B}}, l_{\mathcal{C}})} (b, c).$$

- For any objects b, b' in \mathcal{B} and c, c' in \mathcal{C} , the arrow $s_{(b,c),(b',c')}$ is defined as

$$(b, c) \otimes (b', c') = (b \otimes b', c \otimes c') \xrightarrow{(s_{b,b'}, s_{c,c'})} (b' \otimes b, c' \otimes c) = (b', c') \otimes (b, c).$$

This symmetric monoidal structure, which we still write $\mathcal{B} \times \mathcal{C}$, is also the unique one on the mere category $\mathcal{B} \times \mathcal{C}$ that makes the two projections $p_{\mathcal{B}} : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{B}$ and $p_{\mathcal{C}} : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{C}$ into symmetric strict monoidal functors.

Given any symmetric monoidal functors $(F, F^0, F^2) : \mathcal{A} \rightarrow \mathcal{B}$ and $(G, G^0, G^2) : \mathcal{A} \rightarrow \mathcal{C}$, writing $\langle F, G \rangle$ for the corresponding functor $\mathcal{A} \rightarrow \mathcal{B} \times \mathcal{C}$ via the isomorphism 3.2, this one admits a symmetric monoidal structure as follows:

- Given objects a, a' in \mathcal{A} , $\langle F, G \rangle_{a, a'}^2$ is the arrow of $\mathcal{B} \times \mathcal{C}$

$$\begin{array}{c}
\langle F, G \rangle (a) \otimes \langle F, G \rangle (a') \\
\parallel \\
(F(a), G(a)) \otimes (F(a'), G(a')) \\
\parallel \\
(F(a) \otimes F(a'), G(a) \otimes G(a')) \\
\downarrow (F_{a, a'}^2, G_{a, a'}^2) \\
(F(a \otimes a'), G(a \otimes a')) \\
\parallel \\
\langle F, G \rangle (a \otimes a');
\end{array}$$

- $\langle F, G \rangle^0$ is the arrow

$$I \equiv (I_{\mathcal{B}}, I_{\mathcal{C}}) \xrightarrow{(F^0, G^0)} (F(I_{\mathcal{A}}), G(I_{\mathcal{A}})) \equiv \langle F, G \rangle (I_{\mathcal{A}}).$$

This symmetric monoidal structure is the unique one on the functor $\langle F, G \rangle$ such that the two equalities

$$p_{\mathcal{B}} \circ (\langle F, G \rangle, \langle F, G \rangle^0, \langle F, G \rangle^2) = (F, F^0, F^2)$$

and

$$p_{\mathcal{C}} \circ (\langle F, G \rangle, \langle F, G \rangle^0, \langle F, G \rangle^2) = (G, G^0, G^2)$$

hold in SMC . When talking of $\langle F, G \rangle$ as a symmetric monoidal functor, we will always consider this structure. Note that it is strict if and only both (F, F^0, F^2) and (G, G^0, G^2) are.

Given a pair of monoidal natural transformations between symmetric functors $\sigma : F \rightarrow F' : \mathcal{A} \rightarrow \mathcal{B}$ and $\tau : G \rightarrow G' : \mathcal{A} \rightarrow \mathcal{C}$, the natural transformation $\langle \sigma, \tau \rangle : \langle F, G \rangle \rightarrow \langle F', G' \rangle : \mathcal{A} \rightarrow \mathcal{B} \times \mathcal{C}$ that corresponds to this pair by 3.2 is also monoidal.

The remarks above are enough to show that for any symmetric monoidal categories \mathcal{B} and \mathcal{C} , the isomorphism 3.2 induces a 2-natural isomorphism between 2-functors

3.3

$$SMC(-, \mathcal{B} \times \mathcal{C}) \cong SMC(-, \mathcal{B}) \times SMC(-, \mathcal{C}) : SMC \rightarrow \mathbf{Cat}.$$

This isomorphism in \mathbf{Cat} is also 2-natural in \mathcal{B} and \mathcal{C} for a unique 2-functor $- \times - : SMC \times SMC \rightarrow SMC$. This 2-functor acts on 1-cells as follows. Given any symmetric monoidal functors $(F, F^0, F^2) : \mathcal{B} \rightarrow \mathcal{B}'$ and any $(G, G^0, G^2) : \mathcal{C} \rightarrow \mathcal{C}'$, the monoidal structure on $F \times G : \mathcal{B} \times \mathcal{C} \rightarrow \mathcal{B}' \times \mathcal{C}'$, is the following:

- The arrow $(F \times G)^0$ is

$$I_{\mathcal{B} \times \mathcal{C}} \equiv (I_{\mathcal{B}}, I_{\mathcal{C}}) \xrightarrow{(F^0, G^0)} (F(I_{\mathcal{B}'}), G(I_{\mathcal{C}'})) \equiv (F \times G)(I_{\mathcal{B}'}, I_{\mathcal{C}'}) \equiv (F \times G)(I_{\mathcal{B}' \times \mathcal{C}'});$$

- For any objects b, b' in \mathcal{B} and c, c' in \mathcal{C} , the arrow $(F \times G)_{(b,c),(b',c')}^2$ is

$$\begin{array}{c}
(F \times G)(b, c) \otimes (F \times G)(b', c') \\
\parallel \\
(Fb \otimes Fb', Gc \otimes Gc') \\
\downarrow (F_{b,b'}^2, G_{c,c'}^2) \\
(F(b \otimes b'), G(c \otimes c')) \\
\parallel \\
(F \times G)((b, c) \otimes (b', c')).
\end{array}$$

Note that for any symmetric monoidal category \mathcal{A} the diagonal functor $\mathcal{A} \rightarrow \mathcal{A} \times \mathcal{A}$ admits a strict symmetric monoidal structure by the above results.

4 Tensor as a symmetric monoidal functor

For any symmetric monoidal category \mathcal{A} , its tensor $Ten : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{A}$ admits the following symmetric monoidal structure.

- For any objects a, b, c and d in \mathcal{A} , the arrow $Ten_{(a,b),(c,d)}^2$ is the canonical arrow commuting the b and c

$$\begin{array}{c}
Ten(a, b) \otimes Ten(c, d) \\
\parallel \\
(a \otimes b) \otimes (c \otimes d) \\
\downarrow \\
(a \otimes c) \otimes (b \otimes d) \\
\parallel \\
Ten(a \otimes c, b \otimes d) \\
\parallel \\
Ten((a, b) \otimes (c, d)).
\end{array}$$

- The arrow $Ten^0 : I_{\mathcal{A}} \rightarrow I_{\mathcal{A}} \otimes I_{\mathcal{A}}$ is also the canonical one.

The collection of arrows $Ten_{(a,b),(c,d)}^2$ is natural in (a, b) and (c, d) and Axioms 2.6, 2.7, 2.8, and 2.9 for the functor Ten all amount to commutations of canonical diagrams in \mathcal{A} .

Observe then that the natural isomorphisms part of the monoidal structure of \mathcal{A} , namely ass ,

r , l and s are *monoidal*, precisely they are the respective following 2-cells in SMC

$$\begin{array}{ccc}
\mathcal{A} \times (\mathcal{A} \times \mathcal{A}) & \xrightarrow{\cong} & (\mathcal{A} \times \mathcal{A}) \times \mathcal{A} \\
1 \times Ten \downarrow & \nearrow ass & \downarrow Ten \times 1 \\
\mathcal{A} \times \mathcal{A} & & \mathcal{A} \times \mathcal{A} \\
& \searrow Ten & \swarrow Ten \\
& \mathcal{A} &
\end{array}$$

$$\begin{array}{ccc}
\mathcal{A} \times 1 & \xleftarrow{\cong} & \mathcal{A} \\
1 \times \Delta_I \downarrow & \nearrow r & \parallel \\
\mathcal{A} \times \mathcal{A} & \xrightarrow{Ten} & \mathcal{A}
\end{array}
\qquad
\begin{array}{ccc}
1 \times \mathcal{A} & \xleftarrow{\cong} & \mathcal{A} \\
\Delta_I \times 1 \downarrow & \nearrow l & \parallel \\
\mathcal{A} \times \mathcal{A} & \xrightarrow{Ten} & \mathcal{A}
\end{array}$$

$$\begin{array}{ccc}
\mathcal{A} \times \mathcal{A} & \xrightarrow{Ten} & \mathcal{A} \\
rev \downarrow & \nearrow s & \parallel \\
\mathcal{A} \times \mathcal{A} & \xrightarrow{Ten} & \mathcal{A}
\end{array}$$

where rev stands for the automorphism of $\mathcal{A} \times \mathcal{A}$ in SMC such that $p_1 \circ rev = p_2$ and $p_2 \circ rev = p_1$ for the two projections $p_1, p_2 : \mathcal{A} \otimes \mathcal{A} \rightarrow \mathcal{A}$. All this results straightforwardly again from the coherence in \mathcal{A} .

5 Internal homs in SMC

Given arbitrary symmetric monoidal categories \mathcal{A} and \mathcal{B} , the category $SMC(\mathcal{A}, \mathcal{B})$ of symmetric monoidal functors $\mathcal{A} \rightarrow \mathcal{B}$ and monoidal transformations between them admits a symmetric monoidal structure denoted $[\mathcal{A}, \mathcal{B}]$ that is described below.

The unit $I_{[\mathcal{A}, \mathcal{B}]}$ is given by the constant functor to the unit $I_{\mathcal{B}}$ of \mathcal{B} with the (strong) monoidal structure as described previously.

For any symmetric monoidal functors $F, G : \mathcal{A} \rightarrow \mathcal{B}$, their tensor $F \square G$ is the composite in SMC

$$\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \times \mathcal{A} \xrightarrow{F \times G} \mathcal{B} \times \mathcal{B} \xrightarrow{Ten} \mathcal{B}$$

where Δ is the diagonal functor of \mathcal{A} and Ten the tensor in \mathcal{B} . Therefore $(F \square G)^0$ is the arrow

$$I_{\mathcal{B}} \xrightarrow{\cong} I_{\mathcal{B}} \otimes I_{\mathcal{B}} \xrightarrow{F^0 \otimes G^0} F(I_{\mathcal{A}}) \otimes G(I_{\mathcal{A}}) = F \square G(I_{\mathcal{A}})$$

and $(F \square G)^2$ has component in any pair (a, b) the arrow

$$\begin{array}{c}
F \square G(a) \otimes F \square G(b) \\
\parallel \\
(F(a) \otimes G(a)) \otimes (F(b) \otimes G(b)) \\
\downarrow \cong \\
(F(a) \otimes F(b)) \otimes (G(a) \otimes G(b)) \\
\downarrow F_{a,b}^2 \otimes G_{a,b}^2 \\
F(a \otimes b) \otimes G(a \otimes b) \\
\parallel \\
F \square G(a \otimes b).
\end{array}$$

One defines for any two 2-cells $\sigma : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$ and $\tau : F' \rightarrow G' : \mathcal{A} \rightarrow \mathcal{B}$ in SMC the 2-cell $\sigma \square \tau : F \square G \rightarrow F' \square G'$ in SMC as

$$\begin{array}{ccccc}
& & F \times G & & \\
& \swarrow & \parallel & \searrow & \\
\mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \times \mathcal{A} & \xrightarrow{\sigma \times \tau} & \mathcal{B} \times \mathcal{B} \xrightarrow{Ten} \mathcal{B}, \\
& \nwarrow & \parallel & \nearrow & \\
& & F' \times G' & &
\end{array}$$

which means that for any object a of \mathcal{A} , $(\sigma \square \tau)_a = \sigma_a \otimes \tau_a$.

That the assignments \square define a bifunctor $SMC(\mathcal{A}, \mathcal{B}) \times SMC(\mathcal{A}, \mathcal{B}) \rightarrow SMC(\mathcal{A}, \mathcal{B})$ results from the fact that the cartesian product defines a 2-functor $SMC \times SMC \rightarrow SMC$.

For any symmetric monoidal $F, G, H : \mathcal{A} \rightarrow \mathcal{B}$, the 2-cell $ass_{F,G,H}$ in SMC is

$$\begin{array}{ccccc}
& & \mathcal{A} & & \\
& & \downarrow \Delta & & \\
& & \mathcal{A} \times \mathcal{A} & & \\
& \swarrow 1 \times \Delta & & \searrow \Delta \times 1 & \\
\mathcal{A} \times (\mathcal{A} \times \mathcal{A}) & \xrightarrow{\cong} & (\mathcal{A} \times \mathcal{A}) \times \mathcal{A} & & \\
\downarrow F \times (G \times H) & & \downarrow (F \times G) \times H & & \\
\mathcal{B} \times (\mathcal{B} \times \mathcal{B}) & \xrightarrow{\cong} & (\mathcal{B} \times \mathcal{B}) \times \mathcal{B} & & \\
\downarrow 1 \times Ten & \nearrow ass_{\mathcal{B}} & \downarrow Ten \times 1 & & \\
\mathcal{B} \times \mathcal{B} & & \mathcal{B} \times \mathcal{B} & & \\
& \searrow Ten & \swarrow Ten & & \\
& \mathcal{B}. & & &
\end{array}$$

For any symmetric monoidal $F : \mathcal{A} \rightarrow \mathcal{B}$, the 2-cell r_F in SMC is

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{B} \\ \Delta \downarrow & & \cong \uparrow & \nearrow r & \uparrow Ten \\ \mathcal{A} \times \mathcal{A} & \xrightarrow{F \times !} & \mathcal{B} \times \mathbf{1} & \xrightarrow{1 \times \Delta_{I_{\mathcal{B}}}} & \mathcal{B} \times \mathcal{B} \end{array}$$

and l_F is

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{F} & \mathcal{B} & \xrightarrow{\quad} & \mathcal{B} \\ \Delta \downarrow & & \cong \uparrow & \nearrow l & \uparrow Ten \\ \mathcal{A} \times \mathcal{A} & \xrightarrow{! \times F} & \mathbf{1} \times \mathcal{B} & \xrightarrow{\Delta_{I_{\mathcal{B}}} \times 1} & \mathcal{B} \times \mathcal{B}. \end{array}$$

For any symmetric monoidal $F, G : \mathcal{A} \rightarrow \mathcal{B}$, the 2-cell $s_{F,G} : F \square G \rightarrow G \square F$ in SMC is

$$\begin{array}{ccccccc} \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \times \mathcal{A} & \xrightarrow{F \times G} & \mathcal{B} \times \mathcal{B} & \xrightarrow{Ten} & \mathcal{B} \\ \parallel & & \downarrow rev & & \downarrow rev & \nearrow s & \parallel \\ \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \times \mathcal{A} & \xrightarrow{G \times F} & \mathcal{B} \times \mathcal{B} & \xrightarrow{Ten} & \mathcal{B}. \end{array}$$

The previous definitions amount to say that the canonical isomorphisms ass , r , l , s for $[\mathcal{A}, \mathcal{B}]$ are “pointwise”. That is, for any object a in \mathcal{A} , the isomorphisms:

- $ass_{F,G,H} : F \square (G \square H) \cong (F \square G) \square H$
- $r_F : F \square I \cong F$
- $l_F : I \square F \cong F$

and

- $s_{F,G} : F \square G \cong G \square F$

are respectively defined at any object a by:

- $ass_{F(a),G(a),H(a)} : Fa \otimes (Ga \otimes Ha) \rightarrow (Fa \otimes Ga) \otimes Ha$
- $r_{F(a)} : F(a) \otimes I \rightarrow F(a)$
- $l_{F(a)} : I \otimes F(a) \rightarrow F(a)$

and

- $s_{F(a),G(a)} : F(a) \otimes G(a) \rightarrow G(a) \otimes F(a)$.

From this observation and the fact that \mathcal{B} is symmetric monoidal, it follows easily that the collections $ass_{F,G,H}$, r_F , l_F and $s_{F,G}$ are natural respectively in F, G, H , in F , in F and in F, G and that these together satisfy Axioms 2.1, 2.2, 2.3, 2.4 and 2.5 with \square as tensor and unit $I_{[\mathcal{A}, \mathcal{B}]}$ as specified.

6 The isomorphism $D : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$

Given any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} , we are going to build up from the classical isomorphism

6.1

$$\mathbf{Cat}(\mathcal{A}, \mathbf{Cat}(\mathcal{B}, \mathcal{C})) \cong \mathbf{Cat}(\mathcal{B}, \mathbf{Cat}(\mathcal{A}, \mathcal{C}))$$

an isomorphism of categories

6.2

$$SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \cong SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}])$$

which turns out to have a strict symmetric monoidal structure

$$D_{\mathcal{A},\mathcal{B},\mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$$

with (symmetric) monoidal inverse $D_{\mathcal{B},\mathcal{A},\mathcal{C}}$. Most of the time we will omit the subscripts for D .

To explain more precisely this result, we need to introduce the following terminology:

- We say that two functors or two natural transformations corresponding via the isomorphism 6.1 are *dual* (to each other);
- We write U (rather than $U_{\mathcal{A},\mathcal{B}}$) for the forgetful functor $SMC(\mathcal{A}, \mathcal{B}) \rightarrow \mathbf{Cat}(\mathcal{A}, \mathcal{B})$;
- For any symmetric monoidal functor $(F, F^0, F^2) : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, we let F^\bullet denote the dual functor $\mathcal{B} \rightarrow \mathbf{Cat}(\mathcal{A}, \mathcal{C})$ of the mere functor

$$\mathcal{A} \xrightarrow{F} SMC(\mathcal{B}, \mathcal{C}) \xrightarrow{U} \mathbf{Cat}(\mathcal{B}, \mathcal{C});$$

- For any monoidal natural transformation $\theta : F \rightarrow G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ between symmetric functors, we let θ^\bullet denote the natural transformation $F^\bullet \rightarrow G^\bullet : \mathcal{B} \rightarrow \mathbf{Cat}(\mathcal{A}, \mathcal{C})$ dual of

$$U * \theta : U \circ F \rightarrow U \circ G : \mathcal{A} \rightarrow \mathbf{Cat}(\mathcal{B}, \mathcal{C}).$$

Then we are going to exhibit for any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, a factorisation of F^\bullet as

6.3

$$\mathcal{B} \xrightarrow{F^*} SMC(\mathcal{A}, \mathcal{C}) \xrightarrow{U} \mathbf{Cat}(\mathcal{A}, \mathcal{C})$$

and define on F^* a symmetric monoidal structure $(F^*, F^{*0}, F^{*2}) : \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{C}]$. Also for any monoidal natural transformation between symmetric functors $\theta : (F, F^0, F^2) \rightarrow (G, G^0, G^2) : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, we are going to exhibit a factorisation of $\theta^\bullet : F^\bullet \rightarrow G^\bullet$ as

6.4

$$\begin{array}{ccc} & F^* & \\ \curvearrowright & \parallel & \curvearrowleft \\ \mathcal{B} & \xrightarrow{\theta^*} & SMC(\mathcal{A}, \mathcal{C}) \xrightarrow{U} \mathbf{Cat}(\mathcal{A}, \mathcal{C}) \\ \curvearrowleft & \parallel & \curvearrowright \\ & G^* & \end{array}$$

such that the natural θ^* is monoidal $(F^*, F^{*0}, F^{*2}) \rightarrow (G^*, G^{*0}, G^{*2}) : \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{C}]$. Furthermore any symmetric monoidal functor $(F, F^0, F^2) : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ is equal to $(F^{**}, (F^{**})^0, (F^{**})^2)$. Eventually we define the isomorphism 6.2 as the functor sending any F to F^* and any θ to θ^* .

Let us consider any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ with corresponding $F^\bullet : \mathcal{B} \rightarrow \mathbf{Cat}(\mathcal{A}, \mathcal{C})$. For any b , we define the symmetric monoidal structure $(F^*b, (F^*b)^0, (F^*b)^2)$ on $F^\bullet b : \mathcal{A} \rightarrow \mathcal{C}$ (i.e. $F^*b = F^\bullet b$), as follows:

6.5

- $(F^*b)^0 : I_{\mathcal{C}} \rightarrow F^*(b)(I_{\mathcal{A}})$ is the arrow $(F^0)_b : I_{\mathcal{C}} \rightarrow F(I_{\mathcal{A}})(b)$, component in b of the (monoidal) natural transformation $F^0 : I \rightarrow F(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$.
- For any objects a and a' in \mathcal{A} , $(F^*b)_{a,a'}^2 : F^*b(a) \otimes F^*b(a') \rightarrow F^*b(a \otimes a')$ is the arrow

$$((F^2)_{a,a'})_b : Fa(b) \otimes Fa'(b) \rightarrow F(a \otimes a')(b),$$

component in b of the natural transformation $F_{a,a'}^2 : Fa \square Fa' \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}$.

Axiom 2.6 for $(F^*b, (F^*b)^0, (F^*b)^2)$ is that for any objects a, a', a'' in \mathcal{A} , the diagram in \mathcal{C}

$$\begin{array}{ccc}
Fa(b) \otimes (F(a')(b) \otimes F(a'')(b)) & \xrightarrow{ass} & (Fa(b) \otimes F(a')(b)) \otimes F(a'')(b) \\
\downarrow 1 \otimes (F_{a',a''}^2)_b & & \downarrow (F_{a,a'}^2)_b \otimes 1 \\
Fa(b) \otimes (F(a')(b) \otimes F(a'')(b)) & & (Fa(b) \otimes F(a')(b)) \otimes F(a'')(b) \\
\downarrow (F_{a,a' \otimes a''}^2)_b & & \downarrow (F_{a \otimes a', a''}^2)_b \\
F(a \otimes (a' \otimes a''))(b) & \xrightarrow{F(ass)_b} & F((a \otimes a') \otimes a'')(b)
\end{array}$$

commutes. This diagram is the evaluation in b of the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc}
Fa \square (F(a') \square F(a'')) & \xrightarrow{ass} & (Fa \square F(a')) \square F(a'') \\
\downarrow 1 \square F_{a',a''}^2 & & \downarrow F_{a,a'}^2 \square 1 \\
Fa \square (F(a') \square F(a'')) & & (Fa \square F(a')) \square F(a'') \\
\downarrow F_{a,a' \otimes a''}^2 & & \downarrow F_{a \otimes a', a''}^2 \\
F(a \otimes (a' \otimes a'')) & \xrightarrow{F(ass)} & F((a \otimes a') \otimes a'')
\end{array}$$

which commutes according to Axiom 2.6 for the triple (F, F^0, F^2) .

One can check in a similar way that Axioms 2.7, 2.8 and 2.9 for the triple $(F^*b, (F^*b)^0, (F^*b)^2)$ are the pointwise versions respectively of Axioms 2.7, 2.8 and 2.9 for the triple (F, F^0, F^2) .

Similarly the naturality in a, a' of the collection of arrows $(F^*b)_{a,a'}^2$ can be deduced as a pointwise version of the naturality in a, a' of the collection of arrows $F_{a,a'}^2 : Fa \square Fa' \rightarrow F(a \otimes a')$ in $[\mathcal{B}, \mathcal{C}]$.

Given any arrow $f : b \rightarrow b'$ in \mathcal{B} , we show now that the natural transformation $F^\bullet f : F^\bullet(b) \rightarrow F^\bullet(b') : \mathcal{A} \rightarrow \mathcal{C}$ is monoidal $F^*b \rightarrow F^*(b')$. We define F^*f as this last arrow of $SMC(\mathcal{A}, \mathcal{C})$.

Axiom 2.10 for $F^\bullet f$ amounts to the commutation for any objects a and a' in \mathcal{A} of the diagram

$$\begin{array}{ccc}
Fa(b) \otimes F(a')(b) & \xrightarrow{(F_{a,a'}^2)_b} & F(a \otimes a')(b) \\
\downarrow Fa(f) \otimes F(a')(f) & & \downarrow F(a \otimes a')(f) \\
Fa(b') \otimes F(a')(b') & \xrightarrow{(F_{a,a'}^2)_{b'}} & F(a \otimes a')(b')
\end{array}$$

in \mathcal{C} , which does commute according to the naturality of $F_{a,a'}^2 : Fa \square Fa' \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}$.

Axiom 2.11 for $F^\bullet f$ amounts to the commutation of

$$\begin{array}{ccc}
& & F(I_{\mathcal{A}})(b) \\
& \nearrow F^0_b & \downarrow F(I_{\mathcal{A}})(f) \\
I & & \\
& \searrow F^0_{b'} & \downarrow \\
& & F(I_{\mathcal{A}})(b')
\end{array}$$

in \mathcal{C} , which holds by naturality of $F^0 : I \rightarrow F(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$.

So far we have checked the factorisation of mere functors $F^\bullet = U \circ F^*$ as expected in 6.3. The monoidal structure $(F^*)^0$ and $(F^*)^2$ on F^* is the following.

6.6

- The monoidal natural transformation $(F^*)^0 : I \rightarrow F^*(I_{\mathcal{B}}) : \mathcal{A} \rightarrow \mathcal{C}$ has component in any a in \mathcal{A} , the arrow in \mathcal{C}

$$(Fa)^0 : I \rightarrow F(a)(I_{\mathcal{B}}).$$

- For any objects b, b' in \mathcal{B} , the monoidal natural transformation

$$(F^*)^2_{b, b'} : F^*(b) \square F^*(b') \rightarrow F^*(b \otimes b') : \mathcal{A} \rightarrow \mathcal{C}$$

has component in any a in \mathcal{A} , the arrow in \mathcal{C}

$$(Fa)^2_{b, b'} : Fa(b) \otimes Fa(b') \rightarrow Fa(b \otimes b').$$

The naturality $I \rightarrow F^*(I_{\mathcal{B}}) : \mathcal{A} \rightarrow \mathcal{C}$ of $(F^*)^0$ is equivalent to the commutation for any arrow $f : a \rightarrow a'$ of \mathcal{A} of the diagram in \mathcal{C}

$$\begin{array}{ccc} I & \xrightarrow{(Fa)^0} & Fa(I_{\mathcal{B}}) \\ \parallel & & \downarrow F(f)_{I_{\mathcal{B}}} \\ I & \xrightarrow{(Fa')^0} & Fa'(I_{\mathcal{B}}) \end{array}$$

which does commute according to Axiom 2.11 for the monoidal $F(f) : Fa \rightarrow Fa' : \mathcal{B} \rightarrow \mathcal{C}$.

Axiom 2.10 for $(F^*)^0$ amounts to the commutation for any objects a, a' in \mathcal{A} of the diagram in \mathcal{C}

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\cong} & I \\ (Fa)^0 \otimes (Fa')^0 \downarrow & & \downarrow (F(a \otimes a'))^0 \\ Fa(I_{\mathcal{B}}) \otimes Fa'(I_{\mathcal{B}}) & \xrightarrow{(F^2_{a, a'})_{I_{\mathcal{B}}}} & F(a \otimes a')(I_{\mathcal{B}}) \end{array}$$

which commutes according to Axiom 2.11 for the monoidal natural transformation $F^2_{a, a'} : Fa \square Fa' \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}$.

Axiom 2.11 for $(F^*)^0$ amounts to the equality in \mathcal{C} of the arrows $(F^*)^0_{I_{\mathcal{A}}} : I \rightarrow F^*(I_{\mathcal{B}})(I_{\mathcal{A}})$ and $(F^*(I_{\mathcal{B}}))^0$. These two are respectively $F(I_{\mathcal{A}})^0 : I \rightarrow F(I_{\mathcal{A}})(I_{\mathcal{B}})$ and $(F^0)_{I_{\mathcal{B}}}$ which are equal according to Axiom 2.11 for the monoidal natural transformation $F^0 : I \rightarrow F(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$.

For any objects b, b' in \mathcal{B} , the naturality $F^*b \square F^*b' \rightarrow F^*(b \otimes b') : \mathcal{A} \rightarrow \mathcal{C}$ of $(F^*)^2_{b, b'}$ amounts to the commutation for any arrow $f : a \rightarrow a'$ of \mathcal{A} of the diagram in \mathcal{C}

$$\begin{array}{ccc} Fa(b) \otimes Fa(b') & \xrightarrow{(Fa)^2_{b, b'}} & Fa(b \otimes b') \\ F(f)_b \otimes F(f)_{b'} \downarrow & & \downarrow F(f)_{b \otimes b'} \\ Fa'(b) \otimes Fa'(b') & \xrightarrow{(Fa')^2_{b, b'}} & Fa'(b \otimes b') \end{array}$$

which does commute according to Axiom 2.10 for the monoidal $F(f) : Fa \rightarrow Fa' : \mathcal{B} \rightarrow \mathcal{C}$.

For any objects b, b' in \mathcal{B} , Axiom 2.10 for $(F^*)_{b,b'}^2$ amounts to the commutation for any objects a, a' in \mathcal{A} of the diagram in \mathcal{C}

$$\begin{array}{ccc}
(Fa(b) \otimes Fa(b')) \otimes (Fa'(b) \otimes Fa'(b')) & \xrightarrow{(Fa)_{b,b'}^2 \otimes (Fa')_{b,b'}^2} & Fa(b \otimes b') \otimes Fa'(b \otimes b') \\
\cong \downarrow & & \downarrow (F_{a,a'}^2)_{b \otimes b'} \\
(Fa(b) \otimes Fa'(b)) \otimes (Fa(b') \otimes Fa'(b')) & & \\
(F_{a,a'}^2)_b \otimes (F_{a,a'}^2)_{b'} \downarrow & & \downarrow \\
F(a \otimes a')b \otimes F(a \otimes a')b' & \xrightarrow{(F(a \otimes a'))_{b,b'}^2} & F(a \otimes a')(b \otimes b').
\end{array}$$

This diagram commutes according to Axiom 2.10 for the monoidal natural transformation $F_{a,a'}^2 : Fa \square Fa' \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}$.

For any objects b, b' in \mathcal{B} , Axiom 2.11 for $(F^*)_{b,b'}^2$ amounts to the equality for any objects b and b' in \mathcal{B} of the two arrows in \mathcal{C}

$$I \xrightarrow{(F^*b \square F^*b')^0} (F^*b \square F^*b')(I_{\mathcal{A}}) \xrightarrow{((F^*)_{b,b'}^2)_{I_{\mathcal{A}}}} F^*(b \otimes b')(I_{\mathcal{A}})$$

and $(F^*(b \otimes b'))^0 : I \rightarrow F^*(b \otimes b')(I_{\mathcal{A}})$.

The first arrow rewrites

$$I_{\mathcal{C}} \xrightarrow{\cong} I_{\mathcal{C}} \otimes I_{\mathcal{C}} \xrightarrow{F^0_b \otimes F^0_{b'}} F(I_{\mathcal{A}})(b) \otimes F(I_{\mathcal{A}})(b') \xrightarrow{(F(I_{\mathcal{A}}))_{b,b'}^2} F(I_{\mathcal{A}})(b \otimes b')$$

and the second one is $F^0_{b \otimes b'} : I_{\mathcal{C}} \rightarrow F(I_{\mathcal{A}})(b \otimes b')$. These two arrows are equal according to Axiom 2.10 for the monoidal natural transformation $F^0 : I \rightarrow F(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$. (Remember that $I_{[\mathcal{B}, \mathcal{C}]}^2_{b,b'}$ is the canonical isomorphism $I_{\mathcal{C}} \otimes I_{\mathcal{C}} \rightarrow I_{\mathcal{C}}$.)

The naturality in b and b' of the collection of arrows $(F^*)_{b,b'}^2 : F^*b \square F^*b' \rightarrow F^*(b \otimes b')$ in $[\mathcal{A}, \mathcal{C}]$ is the commutation for any arrows $f : b_1 \rightarrow b'_1$ and $g : b_2 \rightarrow b'_2$ in \mathcal{B} of the diagram in $[\mathcal{A}, \mathcal{C}]$

$$\begin{array}{ccc}
F^*b_1 \square F^*b'_1 & \xrightarrow{(F^*)_{b_1,b'_1}^2} & F^*(b_1 \otimes b'_1) \\
F^*f \square F^*g \downarrow & & \downarrow F^*(f \otimes g) \\
F^*b_2 \square F^*b'_2 & \xrightarrow{(F^*)_{b_2,b'_2}^2} & F^*(b_2 \otimes b'_2)
\end{array}$$

which is pointwise in any a

$$\begin{array}{ccc}
Fa(b_1) \otimes Fa(b'_1) & \xrightarrow{(Fa)_{b_1,b'_1}^2} & Fa(b_1 \otimes b'_1) \\
Fa(f) \otimes Fa(g) \downarrow & & \downarrow Fa(f \otimes g) \\
Fa(b_2) \otimes Fa(b'_2) & \xrightarrow{(Fa)_{b_2,b'_2}^2} & Fa(b_2 \otimes b'_2)
\end{array}$$

that does commute according to the naturality in b, b' of the collection of arrows $(Fa)_{b, b'}^2 : Fa(b) \otimes Fa(b') \rightarrow Fa(b \otimes b')$.

Axioms 2.6 for the triple $(F^*, (F^*)^0, (F^*)^2)$ amounts to the commutation for any objects b, b', b'' in \mathcal{B} of the diagram in $[\mathcal{A}, \mathcal{C}]$

$$\begin{array}{ccc}
F^*b \square (F^*b' \square F^*b'') & \xrightarrow{ass} & (F^*b \square F^*b') \square F^*b'' \\
\downarrow 1 \square (F^*)_{b', b''}^2 & & \downarrow (F^*)_{b, b'}^2 \square 1 \\
F^*b \square F^*(b' \otimes b'') & & F^*(b \otimes b') \square F^*b'' \\
\downarrow (F^*)_{b, b' \otimes b''}^2 & & \downarrow (F^*)_{b \otimes b', b''}^2 \\
F^*(b \otimes (b' \otimes b'')) & \xrightarrow{F^*(ass)} & F^*((b \otimes b') \otimes b'').
\end{array}$$

This diagram is pointwise in any a

$$\begin{array}{ccc}
Fa(b) \otimes (Fa(b') \otimes Fa(b'')) & \xrightarrow{ass} & (Fa(b) \otimes Fa(b')) \otimes Fa(b'') \\
\downarrow 1 \otimes (Fa)_{b', b''}^2 & & \downarrow (Fa)_{b, b'}^2 \otimes 1 \\
Fa(b) \otimes Fa(b' \otimes b'') & & Fa(b \otimes b') \otimes Fa(b'') \\
\downarrow (Fa)_{b, b' \otimes b''}^2 & & \downarrow (Fa)_{b \otimes b', b''}^2 \\
Fa(b \otimes (b' \otimes b'')) & \xrightarrow{Fa(ass)} & Fa((b \otimes b') \otimes b'')
\end{array}$$

which commutes according to Axiom 2.6 for the monoidal functor $Fa : \mathcal{B} \rightarrow \mathcal{C}$.

In the same way, Axioms 2.7 and 2.8 and 2.9 for the triple $(F^*, (F^*)^0, (F^*)^2)$ can be deduced respectively from Axioms 2.7 and 2.8 and 2.9 for the monoidal functors Fa for all objects a of \mathcal{A} .

We show now that given any monoidal natural transformation $\theta : F \rightarrow G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, and any object b in \mathcal{B} , the natural transformation $\theta^\bullet_b : F^\bullet b \rightarrow G^\bullet b : \mathcal{A} \rightarrow \mathcal{C}$ is monoidal $(F^*b, (F^*b)^0, (F^*b)^2) \rightarrow (G^*b, (G^*b)^0, (G^*b)^2)$.

Axiom 2.10 for θ^\bullet_b amounts to the commutation for any objects a, a' in \mathcal{A} of the diagram in \mathcal{C}

$$\begin{array}{ccc}
Fa(b) \otimes Fa'(b) & \xrightarrow{(F^2_{a, a'})_b} & F(a \otimes a')(b) \\
(\theta_a)_b \otimes (\theta_{a'})_b \downarrow & & \downarrow (\theta_{a \otimes a'})_b \\
Ga(b) \otimes Ga'(b) & \xrightarrow{(G^2_{a, a'})_b} & G(a \otimes a')(b)
\end{array}$$

which is the pointwise version in b of

$$\begin{array}{ccc}
Fa \square Fa' & \xrightarrow{F^2_{a, a'}} & F(a \otimes a') \\
\theta_a \square \theta_{a'} \downarrow & & \downarrow \theta_{a \otimes a'} \\
Ga \square Ga' & \xrightarrow{G^2_{a, a'}} & G(a \otimes a')
\end{array}$$

in $[\mathcal{B}, \mathcal{C}]$, that commutes according to Axiom 2.10 for $\theta : F \rightarrow G$. Similarly Axiom 2.11 for θ_b^\bullet is the pointwise version in b of Axiom 2.11 for θ .

Therefore the natural transformation $\theta^\bullet : F^\bullet \rightarrow G^\bullet$ factorises as $U * \theta^*$ as claimed in 6.4. It remains to show that θ^* is monoidal $F^* \rightarrow G^* : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$. Axiom 2.10 for θ^* is that for any objects b and b' in \mathcal{B} the diagram in $[\mathcal{A}, \mathcal{C}]$ commutes

$$\begin{array}{ccc} F^*b \square F^*b' & \xrightarrow{(F^*)_{b,b'}^2} & F^*(b \otimes b') \\ \theta^*_b \square \theta^*_{b'} \downarrow & & \downarrow \theta^*_{b \otimes b'} \\ G^*b \square G^*b' & \xrightarrow{(G^*)_{b,b'}^2} & G^*(b \otimes b'). \end{array}$$

Pointwise in any a in \mathcal{A} , this diagram is

$$\begin{array}{ccc} Fa(b) \otimes Fa(b') & \xrightarrow{(Fa)_{b,b'}^2} & Fa(b \otimes b') \\ (\theta_a)_b \square (\theta_a)_{b'} \downarrow & & \downarrow (\theta_a)_{b \otimes b'} \\ Ga(b) \square Ga(b') & \xrightarrow{(Ga)_{b,b'}^2} & Ga(b \otimes b') \end{array}$$

which commutes according to Axiom 2.10 for the monoidal $\theta_a : Fa \rightarrow Ga : \mathcal{B} \rightarrow \mathcal{C}$. Similarly Axiom 2.11 for θ^* can be checked pointwise and results from Axiom 2.11 for the monoidal transformations $\theta_a : Fa \rightarrow Ga$ for all objects a of \mathcal{A} .

It is now straightforward to check that the isomorphism 6.2 admits a strict monoidal structure $D_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$. We detail briefly below the computation to establish the equality $(F \square G)^* = F^* \square G^* : \mathcal{B} \rightarrow \mathcal{C}$ for any symmetric monoidal functors $F, G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$.

For any object b of \mathcal{B} , the functors $(F^* \square G^*)(b)$ and $(F \square G)^*(b)$ from \mathcal{A} to \mathcal{C} both send any arrow $f : a \rightarrow a'$ to the arrow

$$F(f)_b \otimes G(f)_b : F(a)(b) \otimes G(a)(b) \rightarrow F(a')(b) \otimes G(a')(b)$$

in \mathcal{C} .

For any objects b in \mathcal{B} and a, a' in \mathcal{A} , the arrows $((F \square G)^*(b))_{a,a'}^2$ and $((F^* \square G^*)(b))_{a,a'}^2$ in \mathcal{C} are both equal to

$$\begin{array}{c} (F(a)(b) \otimes G(a)(b)) \otimes (F(a')(b) \otimes G(a')(b)) \\ \downarrow \cong \\ (F(a)(b) \otimes F(a')(b)) \otimes (G(a)(b) \otimes G(a')(b)) \\ \downarrow (F_{a,a'}^2)_b \otimes (G_{a,a'}^2)_b \\ F(a \otimes a')(b) \otimes G(a \otimes a')(b). \end{array}$$

For any object b in \mathcal{B} , the arrows $((F \square G)^*(b))^0$ and $((F^* \square G^*)(b))^0$ in \mathcal{C} are both equal to

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{F_b^0 \otimes G_b^0} F(I_{\mathcal{A}})(b) \otimes G(I_{\mathcal{A}})(b).$$

For any arrow $g : b \rightarrow b'$ in \mathcal{B} , $(F \square G)^*$ and $F^* \square G^*$ both send g to the transformation between functors $\mathcal{A} \rightarrow \mathcal{C}$ with component in any object a , the arrow

$$F(a)(g) \otimes G(a)(g) : F(a)(b) \otimes G(a)(b) \rightarrow F(a)(b') \otimes G(a)(b').$$

For any objects b, b' in \mathcal{B} , the natural transformations between functors $\mathcal{A} \rightarrow \mathcal{C}$

$$((F \square G)^*)_{b, b'}^2 : (F \square G)^*(b) \square (F \square G)^*(b') \rightarrow (F \square G)^*(b \otimes b')$$

and

$$(F^* \square G^*)_{b, b'}^2 : (F^*(b) \square G^*(b)) \square (F^*(b') \square G^*(b')) \rightarrow F^*(b \otimes b') \square G^*(b \otimes b')$$

have both for component in any a the arrow

$$\begin{array}{c} (F(a)(b) \otimes G(a)(b)) \otimes (F(a)(b') \otimes G(a)(b')) \\ \downarrow \cong \\ (F(a)(b) \otimes F(a)(b')) \otimes (G(a)(b) \otimes G(a)(b')) \\ \downarrow (F a)_{b, b'}^2 \otimes (G a)_{b, b'}^2 \\ F(a)(b \otimes b') \otimes G(a)(b \otimes b'). \end{array}$$

The natural transformations between functors $\mathcal{A} \rightarrow \mathcal{C}$

$$((F \square G)^*)^0 : I_{[\mathcal{A}, \mathcal{C}]} \rightarrow (F \square G)^*(I_{\mathcal{B}})$$

and

$$(F^* \square G^*)^0 : I_{[\mathcal{A}, \mathcal{C}]} \rightarrow (F^* \square G^*)(I_{\mathcal{B}})$$

have both for component in any object a the arrow of \mathcal{C}

$$I \xrightarrow{\cong} I \otimes I \xrightarrow{F(a)^0 \otimes G(a)^0} F(a)(I_{\mathcal{B}}) \otimes G(a)(I_{\mathcal{B}}).$$

The computation to check that the image by the isomorphism 6.2 of the unit $I : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ is $I : \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{C}]$ is straightforward and left to the reader. That the functor 6.2 preserves the canonical arrows ass , r , l and s is immediate since these arrows are defined “pointwise”.

Eventually to establish that $D_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ has inverse $D_{\mathcal{B}, \mathcal{A}, \mathcal{C}}$ in SMC , the only non immediate point to check is that any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ is equal to F^{**} . To show this for any such F , it remains to check the equality of the monoidal structures of Fa and $F^{**}a$ for all objects a of \mathcal{A} , and then to check the equality of the monoidal structures of F and F^{**} . Those result from the equalities 6.5 and 6.6:

- For any objects a in \mathcal{A} and b, b' in \mathcal{B} , $(F^{**}a)_{b, b'}^2 = (F^*_{b, b'})_a^2 = F(a)_{b, b'}^2$;
- For any objects a in \mathcal{A} , $(F^{**}a)^0 = (F^*a)^0 = (Fa)^0$;
- For any objects a, a' in \mathcal{A} and b in \mathcal{B} , $((F^{**})_{a, a'}^2)_b = (F^*b)_{a, a'}^2 = (F^2_{a, a'})_b$;
- For any object b in \mathcal{B} , $(F^{**})^0_b = F^*(b)^0 = F^0_b$.

Remark 6.7 Any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ is strict (respectively strong) if and only if for all objects b of \mathcal{B} the functors $F^*(b) : \mathcal{A} \rightarrow \mathcal{C}$ are strict (respectively strong).

From now on, we also call *dual* the symmetric monoidal functors or monoidal transformations that correspond via the isomorphism D .

7 F^0 and F^2 are monoidal

It is proved in this section that for any *symmetric* monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$, the natural transformations $F^0 : \Delta_{I_{\mathcal{B}}} \rightarrow F \circ \Delta_{I_{\mathcal{A}}} : \mathbf{1} \rightarrow \mathcal{B}$ and $F^2 : \text{Ten} \circ (F \times F) \rightarrow F \circ \text{Ten} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$ are monoidal.

Lemma 7.1 *For any symmetric monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$, any objects x, y, z, t in \mathcal{A} , the following diagram in \mathcal{B} commutes*

$$\begin{array}{ccc}
 (Fx \otimes Fy) \otimes (Fz \otimes Ft) & \xrightarrow{\cong} & (Fx \otimes Fz) \otimes (Fy \otimes Ft) \\
 \downarrow F^2_{x,y} \otimes F^2_{z,t} & & \downarrow F^2_{x,z} \otimes F^2_{y,t} \\
 F(x \otimes y) \otimes F(z \otimes t) & & F(x \otimes z) \otimes F(y \otimes t) \\
 \downarrow F^2_{x \otimes y, z \otimes t} & & \downarrow F^2_{x \otimes z, y \otimes t} \\
 F((x \otimes y) \otimes (z \otimes t)) & \xrightarrow{F(\cong)} & F((x \otimes z) \otimes (y \otimes t)).
 \end{array}$$

(Note: In the above lemma we omit to say that the two isomorphisms \cong are the canonical ones permuting respectively the Fy , Fz and y and z .)

PROOF: Consider the following pasting of diagrams in \mathcal{B}

$$\begin{array}{ccccccc}
 (Fx \otimes Fy) \otimes (Fz \otimes Ft) & \xrightarrow{F^2 \otimes 1} & F(x \otimes y) \otimes (Fz \otimes Ft) & \xrightarrow{1 \otimes F^2} & F(x \otimes y) \otimes F(z \otimes t) & \xrightarrow{F^2} & F((x \otimes y) \otimes (z \otimes t)) \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 ((Fx \otimes Fy) \otimes Fz) \otimes Ft & \xrightarrow{(F^2 \otimes 1) \otimes 1} & (F(x \otimes y) \otimes Fz) \otimes Ft & \xrightarrow{F^2 \otimes 1} & F((x \otimes y) \otimes z) \otimes Ft & \xrightarrow{F^2} & F(((x \otimes y) \otimes z) \otimes t) \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 (Fx \otimes (Fy \otimes Fz)) \otimes Ft & \xrightarrow{(1 \otimes F^2) \otimes 1} & (Fx \otimes F(y \otimes z)) \otimes Ft & \xrightarrow{F^2 \otimes 1} & F(x \otimes (y \otimes z)) \otimes Ft & \xrightarrow{F^2} & F((x \otimes (y \otimes z)) \otimes t).
 \end{array}$$

(a) (b) (c) (d)

Here diagram (a) commutes since \mathcal{B} is monoidal, diagrams (b) and (c) commute according to Axiom 2.6 for F and diagram (d) commutes by naturality of F^2 . This shows that the diagram (1):

$$\begin{array}{ccccccc}
 (Fx \otimes Fy) \otimes (Fz \otimes Ft) & \xrightarrow{F^2 \otimes 1} & F(x \otimes y) \otimes (Fz \otimes Ft) & \xrightarrow{1 \otimes F^2} & F(x \otimes y) \otimes F(z \otimes t) & \xrightarrow{F^2} & F((x \otimes y) \otimes (z \otimes t)) \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 (Fx \otimes (Fy \otimes Fz)) \otimes Ft & \xrightarrow{(1 \otimes F^2) \otimes 1} & (Fx \otimes F(y \otimes z)) \otimes Ft & \xrightarrow{F^2 \otimes 1} & F(x \otimes (y \otimes z)) \otimes Ft & \xrightarrow{F^2} & F((x \otimes (y \otimes z)) \otimes t)
 \end{array}$$

commutes. Diagram (1) with y and z inverted is the commuting diagram (2):

$$\begin{array}{ccccccc}
 (Fx \otimes (Fz \otimes Fy)) \otimes Ft & \xrightarrow{(1 \otimes F^2) \otimes 1} & (Fx \otimes F(z \otimes y)) \otimes Ft & \xrightarrow{F^2 \otimes 1} & F(x \otimes (z \otimes y)) \otimes Ft & \xrightarrow{F^2} & F((x \otimes (z \otimes y)) \otimes t) \\
 \downarrow \cong & & \downarrow \cong & & \downarrow \cong & & \downarrow \cong \\
 (Fx \otimes Fz) \otimes (Fy \otimes Ft) & \xrightarrow{F^2 \otimes 1} & F(x \otimes z) \otimes (Fy \otimes Ft) & \xrightarrow{1 \otimes F^2} & F(x \otimes z) \otimes F(y \otimes t) & \xrightarrow{F^2} & F((x \otimes z) \otimes (y \otimes t)).
 \end{array}$$

Moreover in the following pasting (3):

$$\begin{array}{ccccccc}
(Fx \otimes (Fy \otimes Fz)) \otimes Ft & \xrightarrow{(1 \otimes F^2) \otimes 1} & (Fx \otimes F(y \otimes z)) \otimes Ft & \xrightarrow{F^2 \otimes 1} & F(x \otimes (y \otimes z)) \otimes Ft & \xrightarrow{F^2} & F((x \otimes (y \otimes z)) \otimes t) \\
\cong \downarrow & (e) & \cong \downarrow & (f) & \cong \downarrow & (g) & \cong \downarrow \\
(Fx \otimes (Fz \otimes Fy)) \otimes Ft & \xrightarrow{(1 \otimes F^2) \otimes 1} & (Fx \otimes F(z \otimes y)) \otimes Ft & \xrightarrow{F^2 \otimes 1} & F(x \otimes (z \otimes y)) \otimes Ft & \xrightarrow{F^2} & F((x \otimes (z \otimes y)) \otimes t)
\end{array}$$

diagram (e) commutes since F is symmetric, and diagrams (f) and (g) commute by naturality of F^2 . A pasting of (1), (2) and (3) above gives the expected result. \blacksquare

Lemma 7.2 *For any symmetric monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$, the following diagram in \mathcal{B} commutes*

$$\begin{array}{ccc}
I_{\mathcal{B}} & \xrightarrow{\cong} & I_{\mathcal{B}} \otimes I_{\mathcal{B}} \\
F^0 \downarrow & & \downarrow F^0 \otimes F^0 \\
& & F(I_{\mathcal{A}}) \otimes F(I_{\mathcal{A}}) \\
& & \downarrow F^2_{I_{\mathcal{A}}, I_{\mathcal{A}}} \\
F(I_{\mathcal{A}}) & \xrightarrow{F(\cong)} & F(I_{\mathcal{A}} \otimes I_{\mathcal{A}})
\end{array}$$

PROOF: The above diagram is the following pasting of commutative diagrams (naturality of r in \mathcal{B} and Axiom 2.7 for F)

$$\begin{array}{ccccc}
I_{\mathcal{B}} & \xleftarrow{r} & I_{\mathcal{B}} \otimes I_{\mathcal{B}} & \xrightarrow{F^0 \otimes F^0} & F(I_{\mathcal{A}}) \otimes F(I_{\mathcal{A}}) \\
F^0 \downarrow & & \downarrow F^0 \otimes 1 & \nearrow 1 \otimes F^0 & \downarrow F^2_{I_{\mathcal{A}}, I_{\mathcal{A}}} \\
& & F(I_{\mathcal{A}}) \otimes I_{\mathcal{B}} & & \\
& \nwarrow r & & & \\
F(I_{\mathcal{A}}) & \xleftarrow{F(r)} & F(I_{\mathcal{A}} \otimes I_{\mathcal{A}}) & &
\end{array}$$

Lemma 7.3 *For any symmetric monoidal $F : \mathcal{A} \rightarrow \mathcal{B}$, the natural transformation*

$$F^2 : \text{Ten} \circ (F \times F) \rightarrow F \circ \text{Ten} : \mathcal{A} \times \mathcal{A} \rightarrow \mathcal{B}$$

is monoidal.

PROOF: That F^2 satisfies Axiom 2.10 is to say that the diagram in \mathcal{B} commutes

$$\begin{array}{ccc}
\text{Ten} \circ (F \times F)(x, y) \otimes \text{Ten} \circ (F \times F)(z, t) & \xrightarrow{(F \circ \text{Ten})^2_{(x, y), (z, t)}} & \text{Ten} \circ (F \times F)((x, y) \otimes (z, t)) \\
\parallel & & \parallel \\
(F(x) \otimes F(y)) \otimes (F(z) \otimes F(t)) & & F(x \otimes z) \otimes F(y \otimes t) \\
\downarrow F^2_{x, y} \otimes F^2_{z, t} & & \downarrow F^2_{x \otimes z, y \otimes t} \\
F(x \otimes y) \otimes F(z \otimes t) & & F((x \otimes z) \otimes (y \otimes t)) \\
\parallel & & \parallel \\
F \circ \text{Ten}(x, y) \otimes F \circ \text{Ten}(z, t) & \xrightarrow{(F \circ \text{Ten})^2_{(x, y), (z, t)}} & F \circ \text{Ten}((x, y) \otimes (z, t))
\end{array}$$

commutes for any objects x, y, z, t in \mathcal{A} . Note that $(Ten \circ (F \times F))_{(x,y),(z,t)}^2$ is the arrow

$$(F(x) \otimes F(y)) \otimes (F(z) \otimes F(t)) \xrightarrow{\cong} (F(x) \otimes F(z)) \otimes (F(y) \otimes F(t)) \xrightarrow{F_{x,z}^2 \otimes F_{y,t}^2} F(x \otimes z) \otimes F(y \otimes t)$$

and that $(F \circ Ten)_{(x,y),(z,t)}^2$ is the arrow

$$F(x \otimes y) \otimes F(z \otimes t) \xrightarrow{F_{x \otimes y, z \otimes t}^2} F((x \otimes y) \otimes (z \otimes t)) \xrightarrow{F(\cong)} F((x \otimes z) \otimes (y \otimes t)).$$

Therefore Axiom 2.10 for F^2 is just Lemma 7.1. Axiom 2.11 for F^2 is just Lemma 7.2. \blacksquare

Lemma 7.4 *Given any symmetric monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$, the arrow $F^0 : I_{\mathcal{B}} \rightarrow F(I_{\mathcal{A}})$ is the unique component of a monoidal natural transformation*

$$\Delta_{I_{\mathcal{B}}} \rightarrow F \circ \Delta_{I_{\mathcal{A}}} : \mathbf{1} \rightarrow \mathcal{B}.$$

PROOF: Axiom 2.10 for this natural transformation is Lemma 7.2 and Axiom 2.11 is trivial. \blacksquare

8 pre- and post-composition functors

In this section, \mathcal{A} , \mathcal{B} and \mathcal{C} stand for arbitrary symmetric monoidal categories. Since SMC is a 2-category, any 2-cell $\sigma : F \rightarrow G : \mathcal{B} \rightarrow \mathcal{C}$ in SMC induces a 2-cell in \mathbf{Cat}

$$SMC(\mathcal{A}, \sigma) : SMC(\mathcal{A}, F) \rightarrow SMC(\mathcal{A}, G) : SMC(\mathcal{A}, \mathcal{B}) \rightarrow SMC(\mathcal{A}, \mathcal{C}),$$

and the above assignments $F \mapsto SMC(\mathcal{A}, F)$ and $\sigma \mapsto SMC(\mathcal{A}, \sigma)$ define a functor

$$post_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : SMC(\mathcal{B}, \mathcal{C}) \rightarrow \mathbf{Cat}(SMC(\mathcal{A}, \mathcal{B}), SMC(\mathcal{A}, \mathcal{C})).$$

We are going to show that the functor $post_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ factorises as

$$SMC(\mathcal{B}, \mathcal{C}) \xrightarrow{Post_{\mathcal{A}, \mathcal{B}, \mathcal{C}}} SMC([\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]) \xrightarrow{U} \mathbf{Cat}(SMC(\mathcal{A}, \mathcal{B}), SMC(\mathcal{A}, \mathcal{C}))$$

and that the above functor $Post_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ admits a symmetric strict monoidal structure $[\mathcal{B}, \mathcal{C}] \rightarrow [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$, denoted $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}}$. The dual of $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}}$, denoted

$$[-, \mathcal{C}]_{\mathcal{A}, \mathcal{B}} : [\mathcal{A}, \mathcal{B}] \rightarrow [[\mathcal{B}, \mathcal{C}], [\mathcal{A}, \mathcal{C}]],$$

will be also briefly considered.

Given any symmetric monoidal functor $F : \mathcal{B} \rightarrow \mathcal{C}$, the monoidal structure

$$[\mathcal{A}, F] = (SMC(\mathcal{A}, F), [\mathcal{A}, F]^0, [\mathcal{A}, F]^2) : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{C}]$$

on the functor $SMC(\mathcal{A}, F)$, is as follows.

The monoidal natural transformation $[\mathcal{A}, F]^0 : I \rightarrow F \circ I_{[\mathcal{A}, \mathcal{B}]} : \mathcal{A} \rightarrow \mathcal{C}$ is the pasting in SMC

$$\begin{array}{ccccc} \mathcal{A} & \xrightarrow{\quad} & \mathbf{1} & \xrightarrow{\Delta_{I_{\mathcal{B}}}} & \mathcal{B} \\ & & \searrow & \nearrow F^0 & \downarrow F \\ & & & & \mathcal{C} \\ & & \searrow \Delta_{I_{\mathcal{C}}} & & \end{array}$$

where 2-cell F^0 is that of Lemma 7.4. The component of $[\mathcal{A}, F]^0$ in any a in \mathcal{A} , is therefore $F^0 : I_{\mathcal{C}} \rightarrow F(I_{\mathcal{B}})$.

For any symmetric monoidal functors $G, H : \mathcal{A} \rightarrow \mathcal{B}$, the monoidal natural transformation $[\mathcal{A}, F]_{G, H}^2 : (F \circ G) \square (F \circ H) \rightarrow F \circ (G \square H) : \mathcal{A} \rightarrow \mathcal{C}$ is the pasting in SMC

$$\begin{array}{ccccc}
 & & \mathcal{C} \otimes \mathcal{C} & & \\
 & \nearrow^{F \times F} & \parallel & \searrow^{Ten} & \\
 \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \times \mathcal{A} & \xrightarrow{G \times H} & \mathcal{B} \times \mathcal{B} \\
 & \searrow_{Ten} & \parallel & \nearrow_F & \\
 & & \mathcal{B} & & \mathcal{C}
 \end{array}$$

F^2

where the 2-cell F^2 is that of Lemma 7.3. The component of $[\mathcal{A}, F]_{G, H}^2$ in any a in \mathcal{A} , is therefore

$$F_{G(a), H(a)}^2 : FG(a) \otimes FH(a) \rightarrow F(G(a) \otimes H(a)).$$

The naturalities in G and H of the collection of arrows $[\mathcal{A}, F]_{G, H}^2 : (F \circ G) \square (F \circ H) \rightarrow F \circ (G \square H)$ of $SMC(\mathcal{A}, \mathcal{C})$, results actually from the fact that SMC is a 2-category. For instance the naturality in G means that for any monoidal transformation $\tau : G \rightarrow G' : \mathcal{A} \rightarrow \mathcal{B}$, the diagram

$$\begin{array}{ccc}
 (F \circ G) \square (F \circ H) & \xrightarrow{(F * \tau) \square 1} & (F \circ G') \square (F \circ H) \\
 \downarrow [\mathcal{A}, F]_{G, H}^2 & & \downarrow [\mathcal{A}, F]_{G', H}^2 \\
 F \circ (G \square H) & \xrightarrow{F * (\tau \square 1)} & F \circ (G' \square H)
 \end{array}$$

in $[\mathcal{A}, \mathcal{C}]$ is commutative and this holds by the interchange law in **Cat** for the composition of the 2-cells

$$\begin{array}{ccccc}
 & & G \times H & & \\
 & \nearrow & \parallel & \searrow & \\
 \mathcal{A} & \xrightarrow{\Delta} & \mathcal{A} \times \mathcal{A} & \xrightarrow{G \times H} & \mathcal{B} \times \mathcal{B} \\
 & \searrow & \parallel & \nearrow & \\
 & & G' \times H & & \mathcal{C} \times \mathcal{C} \\
 & & \parallel & & \parallel \\
 & & \tau \times 1 & & F^2 \\
 & & \parallel & & \parallel \\
 & & G' \times H & & \mathcal{B}
 \end{array}$$

F^2

Axiom 2.6 for the triple $(SMC(\mathcal{A}, F), [\mathcal{A}, F]^2, [\mathcal{A}, F]^0)$ is the commutation for any symmetric monoidal functors $H, K, L : \mathcal{A} \rightarrow \mathcal{B}$ of the diagram in $[\mathcal{A}, \mathcal{C}]$

$$\begin{array}{ccc}
 (F \circ H) \square ((F \circ K) \square (F \circ L)) & \xrightarrow{\cong} & ((F \circ H) \square (F \circ K)) \square (F \circ L) \\
 \downarrow 1 \square [\mathcal{A}, F]_{K, L}^2 & & \downarrow [\mathcal{A}, F]_{H, K}^2 \square 1 \\
 (F \circ H) \square (F \circ (K \square L)) & & (F \circ (H \square K)) \square (F \circ L) \\
 \downarrow [\mathcal{A}, F]_{H, K \square L}^2 & & \downarrow [\mathcal{A}, F]_{H \square K, L}^2 \\
 F \circ (H \square (K \square L)) & \xrightarrow{F * ass} & F \circ ((H \square K) \square L).
 \end{array}$$

This diagram is pointwise in any a

$$\begin{array}{ccc}
FHa \otimes (FKa \otimes FLa) & \xrightarrow{\cong} & (FHa \otimes FKa) \otimes FLa \\
\downarrow 1 \otimes F_{Ka, La}^2 & & \downarrow F_{Ha, Ka}^2 \otimes 1 \\
FHa \otimes F(Ka \otimes La) & & F(Ha \otimes Ka) \otimes FLa \\
\downarrow F_{Ha, Ka \otimes La}^2 & & \downarrow F_{Ha \otimes Ka, La}^2 \\
F(Ha \otimes (Ka \otimes La)) & \xrightarrow{F(ass)} & F((Ha \otimes Ka) \otimes La)
\end{array}$$

that commutes according to Axiom 2.6 for F .

Similarly Axioms 2.7, 2.8 and 2.9 for $(SMC(\mathcal{A}, F), [\mathcal{A}, F]^0, [\mathcal{A}, F]^2)$ can be checked pointwise since they hold for F .

Note the following that is immediate from the definitions.

Remark 8.1 *For any symmetric monoidal functor $F : \mathcal{B} \rightarrow \mathcal{C}$, if F is strict (resp. strong) then $[\mathcal{A}, F] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{C}]$ is also strict (resp. strong).*

Consider now a monoidal natural transformation between symmetric monoidal functors $\sigma : F \rightarrow G : \mathcal{B} \rightarrow \mathcal{C}$. We show that the natural transformation

$$SMC(\mathcal{A}, \sigma) : SMC(\mathcal{A}, F) \rightarrow SMC(\mathcal{A}, G) : SMC(\mathcal{A}, \mathcal{B}) \rightarrow SMC(\mathcal{A}, \mathcal{C})$$

is monoidal with respect to the structures $[\mathcal{A}, F]$ and $[\mathcal{A}, G]$. The resulting 2-cell $[\mathcal{A}, F] \rightarrow [\mathcal{A}, G] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{C}]$ in SMC will be later denoted $[\mathcal{A}, \sigma]$.

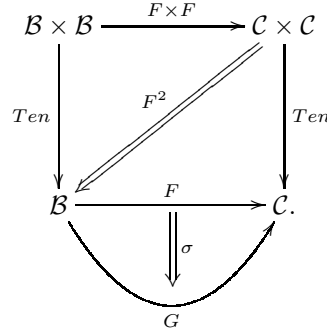
The following pastings in SMC are equal since Axiom 2.10 for σ states their (pointwise) equality

8.2

$$\begin{array}{ccc}
& & F \times F \\
& \searrow & \downarrow \sigma \times \sigma \\
\mathcal{B} \times \mathcal{B} & \xrightarrow{G \times G} & \mathcal{C} \times \mathcal{C} \\
\downarrow Ten & \swarrow G^2 & \downarrow Ten \\
\mathcal{B} & \xrightarrow{G} & \mathcal{C}
\end{array}$$

and

8.3



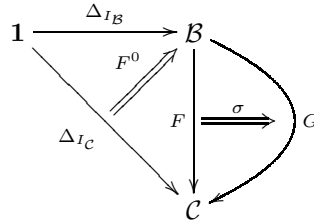
That the natural transformation $SMC(\mathcal{A}, \sigma)$ satisfies Axiom 2.10 for the symmetric monoidal structures $[\mathcal{A}, F]$ and $[\mathcal{A}, G]$, is that for any symmetric monoidal functors $H, K : \mathcal{A} \rightarrow \mathcal{B}$, the diagram below in $[\mathcal{A}, \mathcal{C}]$ commutes

$$\begin{array}{ccc}
 (F \circ H) \square (F \circ K) & \xrightarrow{[\mathcal{A}, F]_{H, K}^2} & F \circ (H \square K) \\
 (\sigma * H) \square (\sigma * K) \downarrow & & \downarrow \sigma * (H \square K) \\
 (G \circ H) \square (G \circ K) & \xrightarrow{[\mathcal{A}, G]_{H, K}^2} & G \circ (H \square K).
 \end{array}$$

This one does commute since the two legs of this diagram are obtained by composing the pastings 8.2 and 8.3 by the arrow $\mathcal{A} \xrightarrow{\Delta} \mathcal{A} \times \mathcal{A} \xrightarrow{H \times K} \mathcal{B} \times \mathcal{B}$.

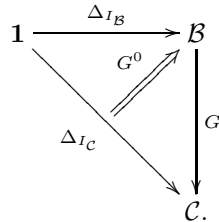
Also the two following pastings in SMC are equal since Axiom 2.11 for σ states their equality

8.4



and

8.5



Axiom 2.11 for $SMC(\mathcal{A}, \sigma)$, $[\mathcal{A}, F]$ and $[\mathcal{A}, G]$, is that the following diagram in $[\mathcal{A}, \mathcal{C}]$ commutes

$$\begin{array}{ccc}
 I_{[\mathcal{A}, \mathcal{C}]} & \xrightarrow{[\mathcal{A}, F]^0} & F \circ I_{[\mathcal{A}, \mathcal{B}]} \\
 \searrow [\mathcal{A}, G]^0 & & \downarrow \sigma * I_{[\mathcal{A}, \mathcal{B}]} \\
 & & G \circ I_{[\mathcal{A}, \mathcal{B}]}
 \end{array}$$

It does since its two legs are obtained by composing the 2-cells 8.4 and 8.5 by the functor $\mathcal{A} \rightarrow \mathbf{1}$.

So far we have exhibited a factorisation of $post_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ as

$$SMC(\mathcal{B}, \mathcal{C}) \xrightarrow{Post} SMC([\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]) \xrightarrow{U} \mathbf{Cat}(SMC(\mathcal{A}, \mathcal{B}), SMC(\mathcal{A}, \mathcal{C})) .$$

It is now rather straightforward to check that the above functor $Post$ admits a strict symmetric monoidal structure $[\mathcal{B}, \mathcal{C}] \rightarrow [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$. This one will be denoted by $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}}$, or simply $[\mathcal{A}, -]$ when no ambiguity can occur. We give a few details of the computation below.

First we check that for any symmetric monoidal functors $F, G : \mathcal{B} \rightarrow \mathcal{C}$, the symmetric monoidal functors $[\mathcal{A}, F \square G]$ and $[\mathcal{A}, F] \square [\mathcal{A}, G] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{C}]$ are equal.

Consider any 2-cell $\sigma : H \rightarrow K : \mathcal{A} \rightarrow \mathcal{B}$ in SMC . According to the 2-naturality of the isomorphism 3.3, the 2-cells

and

are equal in SMC . By composing those with

$$\mathcal{B} \times \mathcal{B} \xrightarrow{F \times G} \mathcal{C} \times \mathcal{C} \xrightarrow{Ten} \mathcal{C} ,$$

one obtains the equality of the images of σ by the functors $[\mathcal{A}, F \square G]$ and $[\mathcal{A}, F] \square [\mathcal{A}, G]$.

For any symmetric monoidal functors $H, K : \mathcal{A} \rightarrow \mathcal{B}$, the arrows in $[\mathcal{A}, \mathcal{C}]$

$$[\mathcal{A}, F \square G]_{H, K}^2 : ((F \square G)H) \square ((F \square G)K) \rightarrow (F \square G)(H \square K)$$

and

$$([\mathcal{A}, F] \square [\mathcal{A}, G])_{H, K}^2 : (FH \square GH) \square (FK \square GK) \rightarrow (F(H \square K)) \square (G(H \square K)),$$

are natural transformations between functors $\mathcal{A} \rightarrow \mathcal{C}$ both with component in any a the arrow

$$(FHa \otimes GHa) \otimes (FKa \otimes GKa) \xrightarrow{\cong} (FHa \otimes FKa) \otimes (GHa \otimes GKa) \xrightarrow{F_{Ha, Ka}^2 \otimes G_{Ha, Ka}^2} F(Ha \otimes Ka) \otimes G(Ha \otimes Ka).$$

The arrows in $[\mathcal{A}, \mathcal{C}]$

$$[\mathcal{A}, F \square G]^0 : I \rightarrow (F \square G) \circ I_{[\mathcal{A}, \mathcal{B}]}$$

and

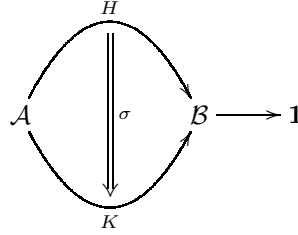
$$([\mathcal{A}, F] \square [\mathcal{A}, G])^0 : I \rightarrow (F \circ I_{[\mathcal{A}, \mathcal{B}]}) \square (G \circ I_{[\mathcal{A}, \mathcal{B}]})$$

are natural transformations both with component in any a the arrow

$$I_{\mathcal{C}} \xrightarrow{\cong} I_{\mathcal{C}} \otimes I_{\mathcal{C}} \xrightarrow{F^0 \otimes G^0} F(I_{\mathcal{B}}) \otimes F(I_{\mathcal{B}}).$$

We check now that the symmetric monoidal functor $[\mathcal{A}, I_{[\mathcal{B}, \mathcal{C}]}]$ is the unit of $[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$.

Consider any 2-cell $\sigma : H \rightarrow K : \mathcal{A} \rightarrow \mathcal{B}$ in SMC . The 2-cell



and the identity 2-cell at the unique arrow $\mathcal{A} \rightarrow \mathbf{1}$ are equal according to the 2-naturality of the isomorphism 3.1. Therefore by composing these 2-cells with the constant functor $\Delta_{I_{\mathcal{C}}} : \mathbf{1} \rightarrow \mathcal{C}$, one obtains the equality between the 2-cells $I_{[\mathcal{B}, \mathcal{C}]} * \sigma$, which is the image σ by the functor $[\mathcal{A}, I_{[\mathcal{B}, \mathcal{C}]}]$, and the identity at $I_{[\mathcal{A}, \mathcal{C}]}$ in $[\mathcal{A}, \mathcal{C}]$, which is the image of σ by the unit of $[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$.

For any symmetric monoidal functors $H, K : \mathcal{A} \rightarrow \mathcal{B}$, the arrows in $[\mathcal{A}, \mathcal{C}]$

$$[\mathcal{A}, I_{[\mathcal{B}, \mathcal{C}]}]_{H, K}^2 : (I_{[\mathcal{B}, \mathcal{C}]} \circ H) \square (I_{[\mathcal{B}, \mathcal{C}]} \circ K) \rightarrow I_{[\mathcal{B}, \mathcal{C}]} \circ (H \square K)$$

and

$$(I_{[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]})_{H, K}^2 : I \square I \rightarrow I$$

are natural transformations both with component in any a , the canonical arrow $I_{\mathcal{C}} \otimes I_{\mathcal{C}} \rightarrow I_{\mathcal{C}}$.

The arrows in $[\mathcal{A}, \mathcal{C}]$

$$[\mathcal{A}, I_{[\mathcal{B}, \mathcal{C}]}]^0 : I \rightarrow I_{[\mathcal{B}, \mathcal{C}]} \circ I_{[\mathcal{A}, \mathcal{B}]}$$

and

$$I_{[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]}^0 : I \rightarrow I$$

are natural transformations both with component in any a , the identity at $I_{\mathcal{C}}$.

Eventually that $[\mathcal{A}, -]$ preserves the canonical arrows ass , r , l and s , results easily from the fact that these arrows are defined pointwise in functor categories. For instance the image by $[\mathcal{A}, -]$ of $ass_{F, G, H} : F \square (G \square H) \rightarrow (F \square G) \square H$, for any symmetric monoidal functors $F, G, H : \mathcal{B} \rightarrow \mathcal{C}$, is the monoidal natural transformation

$$[\mathcal{A}, F \square (G \square H)] \rightarrow [\mathcal{A}, (F \square G) \square H] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{C}]$$

with component in any $K : \mathcal{A} \rightarrow \mathcal{B}$, the monoidal natural transformation

$$ass * K : (F \square (G \square H)) \circ K \rightarrow ((F \square G) \square H) \circ K : \mathcal{A} \rightarrow \mathcal{C},$$

this one has component in any a , the arrow

$$ass : FKa \otimes (GKa \otimes HKa) \rightarrow (FKa \otimes GKa) \otimes HKa.$$

Therefore the component in K of $[\mathcal{A}, ass_{F,G,H}]$ is

$$ass_{FK,GK,HK} : FK \square (GK \square HK) \rightarrow (FK \square GK) \square HK,$$

which is also the component in K of

$$ass_{[\mathcal{A},F],[\mathcal{A},G],[\mathcal{A},H]} : [\mathcal{A}, F] \square ([\mathcal{A}, G] \square [\mathcal{A}, H]) \rightarrow ([\mathcal{A}, F] \square [\mathcal{A}, G]) \square [\mathcal{A}, H].$$

The cases for r , l and s are similar.

Since SMC is a 2-category, one has also the functor

$$pre_{\mathcal{A},\mathcal{B},\mathcal{C}} : SMC(\mathcal{A},\mathcal{B}) \rightarrow \mathbf{Cat}(SMC(\mathcal{B},\mathcal{C}), SMC(\mathcal{A},\mathcal{C}))$$

that assigns any symmetric monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$ to the functor $SMC(F,\mathcal{C}) : SMC(\mathcal{B},\mathcal{C}) \rightarrow SMC(\mathcal{A},\mathcal{C})$ and any monoidal natural transformation $\sigma : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$ to the natural transformation $SMC(\sigma,\mathcal{C}) : SMC(F,\mathcal{C}) \rightarrow SMC(G,\mathcal{C})$. This mere functor pre is the dual to $post : SMC(\mathcal{B},\mathcal{C}) \rightarrow \mathbf{Cat}(SMC(\mathcal{A},\mathcal{B}), SMC(\mathcal{A},\mathcal{C}))$ in the sense of 6.1 and therefore, according to Section 5, it factorises as

$$SMC(\mathcal{A},\mathcal{B}) \xrightarrow{Pre} SMC([\mathcal{B},\mathcal{C}], [\mathcal{A},\mathcal{C}]) \xrightarrow{U} \mathbf{Cat}(SMC(\mathcal{B},\mathcal{C}), SMC(\mathcal{A},\mathcal{C})).$$

for a functor Pre that admits a symmetric monoidal structure $[-,\mathcal{C}]_{\mathcal{A},\mathcal{B}} : [\mathcal{A},\mathcal{B}] \rightarrow [[\mathcal{B},\mathcal{C}], [\mathcal{A},\mathcal{C}]]$ dual in the sense of 6.2 to $[\mathcal{A},-]_{\mathcal{B},\mathcal{C}} : [\mathcal{B},\mathcal{C}] \rightarrow [[\mathcal{A},\mathcal{B}], [\mathcal{A},\mathcal{C}]]$.

Note that according to Remark 6.7, for any symmetric monoidal $F : \mathcal{A} \rightarrow \mathcal{B}$, the symmetric monoidal structure $[-,\mathcal{C}]_{\mathcal{A},\mathcal{B}}(F) : [\mathcal{B},\mathcal{C}] \rightarrow [\mathcal{A},\mathcal{C}]$, which we write $[F,\mathcal{C}]$, is *strict*. Also for any monoidal transformation $\sigma : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$, the natural transformation $SMC(\sigma,\mathcal{C})$ is monoidal $[-,\mathcal{C}](\sigma) : [F,\mathcal{C}] \rightarrow [G,\mathcal{C}]$ and we write $[\sigma,\mathcal{C}]$ for this last 2-cell in SMC .

9 The 2-functor $\underline{Hom} : SMC^{op} \times SMC \rightarrow SMC$

In this section a “hom” functor $SMC^{op} \times SMC \rightarrow SMC$ is defined. It is denoted cautiously \underline{Hom} to avoid ambiguities arising from a possible overuse of the square-brackets notation $[-,-]$.

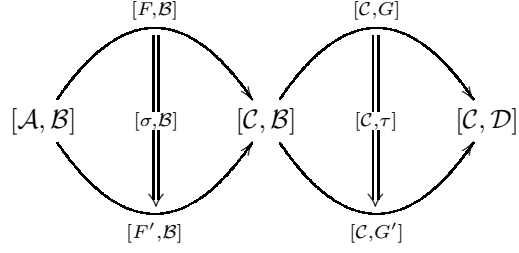
Lemma 9.1 *The diagram in SMC*

$$\begin{array}{ccc} & [\mathcal{A}, \mathcal{B}] & \\ [F,\mathcal{B}] \swarrow & & \searrow [\mathcal{A},G] \\ [\mathcal{C}, \mathcal{B}] & & [\mathcal{A}, \mathcal{D}] \\ [\mathcal{C},G] \searrow & & \swarrow [F,\mathcal{D}] \\ & [\mathcal{C}, \mathcal{D}] & \end{array}$$

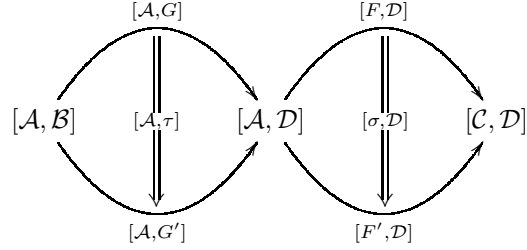
commutes for any $F : \mathcal{C} \rightarrow \mathcal{A}$ and $G : \mathcal{B} \rightarrow \mathcal{D}$.

In SMC , for any 2-cells $\sigma : F \rightarrow F' : \mathcal{C} \rightarrow \mathcal{A}$ and $\tau : G \rightarrow G' : \mathcal{B} \rightarrow \mathcal{D}$ in SMC , the 2-cells of

SMC

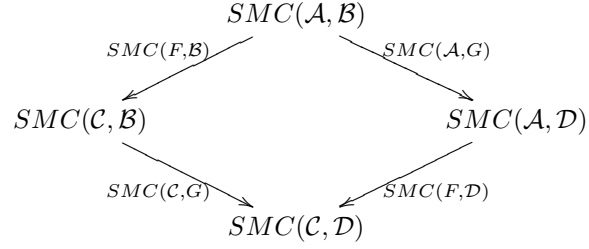


and



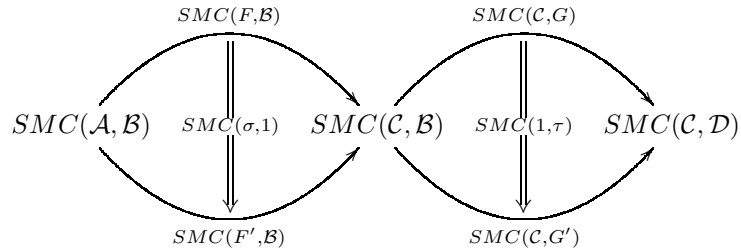
are equal.

PROOF: To show the first assertion, since the following diagram in **Cat** is commutative



it remains to check that the monoidal structures of $[C, G] \circ [F, B]$ and $[F, D] \circ [A, G]$ are the same. Straightforward computations show that this is the case. (The resulting monoidal structure is described below after the definition 9.2).

The assertion regarding the monoidal natural transformations σ and τ follows from the fact that since SMC is a 2-category, the following 2-cells in **Cat**



and

$$\begin{array}{ccccc}
& SMC(\mathcal{A}, G) & & SMC(F, \mathcal{D}) & \\
& \curvearrowright & & \curvearrowright & \\
SMC(\mathcal{A}, \mathcal{B}) & SMC(1, \tau) & SMC(\mathcal{A}, \mathcal{D}) & SMC(\sigma, 1) & SMC(\mathcal{C}, \mathcal{D}) \\
& \curvearrowleft & & \curvearrowleft & \\
& SMC(\mathcal{A}, G') & & SMC(F', \mathcal{D}) &
\end{array}$$

are equal. ■

As a consequence of the previous lemma, we can define for any symmetric monoidal $F : \mathcal{C} \rightarrow \mathcal{A}$ and $G : \mathcal{B} \rightarrow \mathcal{D}$, the symmetric monoidal functor $[F, G] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{D}]$ as either of the composites

9.2 $[\mathcal{C}, G] \circ [F, \mathcal{B}]$ or $[F, \mathcal{D}] \circ [\mathcal{A}, G]$.

Note that the monoidal structure of $[F, G]$ above is the following:

- For any symmetric monoidal functors $H, K : \mathcal{A} \rightarrow \mathcal{B}$, the natural monoidal transformation

$$[F, G]_{H, K}^2 : GHF \square GKF \rightarrow G(H \square K)F : \mathcal{C} \rightarrow \mathcal{D}$$

has components in c , the arrow $G_{HFc, KFc}^2 : GHFc \otimes GKFc \rightarrow G(HFc \otimes KFc)$ in \mathcal{D} .

- The monoidal natural transformation

$$[F, G]^0 : I_{[\mathcal{C}, \mathcal{D}]} \rightarrow G \circ I_{[\mathcal{A}, \mathcal{B}]} \circ F : \mathcal{C} \rightarrow \mathcal{D}$$

has component in any c in \mathcal{C} , the arrow $G^0 : I \rightarrow G(I_{\mathcal{B}})$ in \mathcal{D} .

Also, one can define for any monoidal natural transformations $\sigma : F \rightarrow F' : \mathcal{C} \rightarrow \mathcal{A}$ and $\tau : G \rightarrow G' : \mathcal{B} \rightarrow \mathcal{D}$, the monoidal natural transformation

$$[\sigma, \tau] : [F, G] \rightarrow [F', G'] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{D}]$$

as either of the 2-cells

9.3 $[\mathcal{C}, \tau] * [\sigma, \mathcal{B}]$ or $[\sigma, \mathcal{D}] * [\mathcal{A}, \tau]$.

It follows from this definition that for any symmetric monoidal functors $F, F', F'' : \mathcal{C} \rightarrow \mathcal{A}$ and $G, G', G'' : \mathcal{B} \rightarrow \mathcal{D}$, and any monoidal transformations

$$F \xrightarrow{\sigma} F' \xrightarrow{\sigma'} F'', \text{ and } G \xrightarrow{\tau} G' \xrightarrow{\tau'} G'',$$

the composite

$$[F, G] \xrightarrow{[\sigma, \tau]} [F', G'] \xrightarrow{[\sigma', \tau']} [F'', G''] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{D}]$$

is $[\sigma' \sigma, \tau' \tau]$.

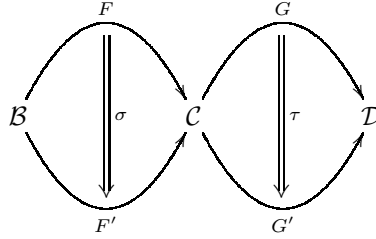
Also for any symmetric monoidal functors $F : \mathcal{C} \rightarrow \mathcal{A}$ and $G : \mathcal{B} \rightarrow \mathcal{D}$, writing 1 for the identity natural transformations at F and G , since the monoidal natural transformations $[1, \mathcal{B}] : [F, \mathcal{B}] \rightarrow [F, \mathcal{B}] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{B}]$ and $[\mathcal{C}, 1] : [\mathcal{C}, G] \rightarrow [\mathcal{C}, G] : [\mathcal{C}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{D}]$ are identities, one has that $[1, 1] : [F, G] \rightarrow [F, G] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{D}]$ is the identity monoidal natural transformation at $[F, G]$.

This is to say the following.

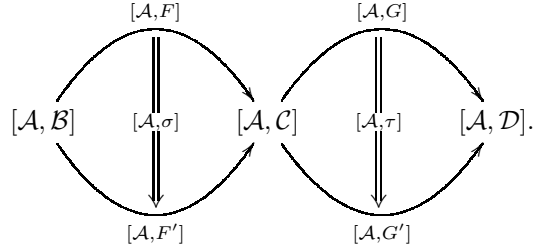
Lemma 9.4 *Given any symmetric monoidal categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} the assignments $(F, G) \mapsto [F, G]$ as in 9.2 and $(\sigma, \tau) \mapsto [\sigma, \tau]$ as in 9.3 define a functor $SMC(\mathcal{C}, \mathcal{A}) \times SMC(\mathcal{B}, \mathcal{D}) \rightarrow SMC([\mathcal{A}, \mathcal{B}], [\mathcal{C}, \mathcal{D}])$.*

Lemma 9.5 *The following statements hold in SMC , for any \mathcal{A} .*

- For any \mathcal{B} , the image of identity 1-cell $1_{\mathcal{B}}$ at \mathcal{B} by $[\mathcal{A}, -]$ is the identity 1-cell at $[\mathcal{A}, \mathcal{B}]$.
- For any 1-cells $F : \mathcal{B} \rightarrow \mathcal{C}$ and $G : \mathcal{C} \rightarrow \mathcal{D}$, the 1-cells $[\mathcal{A}, G \circ F]$ and $[\mathcal{A}, G] \circ [\mathcal{A}, F] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{D}]$ are equal.
- For any 1-cells $\sigma : F \rightarrow F' : \mathcal{B} \rightarrow \mathcal{C}$ and $\tau : G \rightarrow G' : \mathcal{C} \rightarrow \mathcal{D}$, the image by $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{D}}$ of the composite of the 2-cells



is the composite of the 2-cells



PROOF: The functor $SMC(\mathcal{A}, 1_{\mathcal{B}})$ is the identity at the category $SMC(\mathcal{A}, \mathcal{B})$ and the monoidal structure $[\mathcal{A}, 1_{\mathcal{B}}]$ on this functor is strict since the identity $1_{\mathcal{B}}$ is. Therefore $[\mathcal{A}, 1_{\mathcal{B}}]$ is the identity in SMC at $[\mathcal{A}, \mathcal{B}]$.

For the second assertion, since SMC is a 2-category, the mere functors $SMC(\mathcal{A}, G \circ F)$ and $SMC(\mathcal{A}, G) \circ SMC(\mathcal{A}, F) : SMC(\mathcal{A}, \mathcal{B}) \rightarrow SMC(\mathcal{A}, \mathcal{D})$ are equal. It remains to show the equality of the monoidal structures of $[\mathcal{A}, G \circ F]$ and $[\mathcal{A}, G] \circ [\mathcal{A}, F]$. This is again straightforward. For any symmetric monoidal functors $H, K : \mathcal{A} \rightarrow \mathcal{B}$, the monoidal natural transformations $([\mathcal{A}, G] \circ [\mathcal{A}, F])_{H, K}^2$ and $([\mathcal{A}, GF])_{H, K}^2 : GFH \square GFK \rightarrow GF(H \square K) : \mathcal{A} \rightarrow \mathcal{D}$ are equal, with components in any object a in \mathcal{A} , the arrow

$$GFHa \otimes GFKa \xrightarrow{G_{FH a, FKa}^2} G(FHa \otimes FKa) \xrightarrow{G(F_{Ha, Ka}^2)} G(F(Ha \otimes Ka))$$

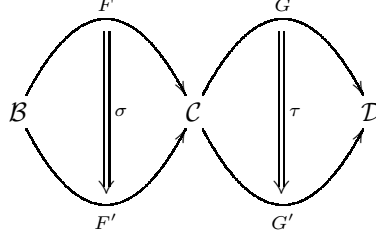
in \mathcal{D} .

The monoidal natural transformations $([\mathcal{A}, G] \circ [\mathcal{A}, F])^0$ and $([\mathcal{A}, GF])^0 : I_{[\mathcal{A}, \mathcal{D}]} \rightarrow GF I_{[\mathcal{A}, \mathcal{B}]}$ are equal, with components in any object a of \mathcal{A} , the arrow

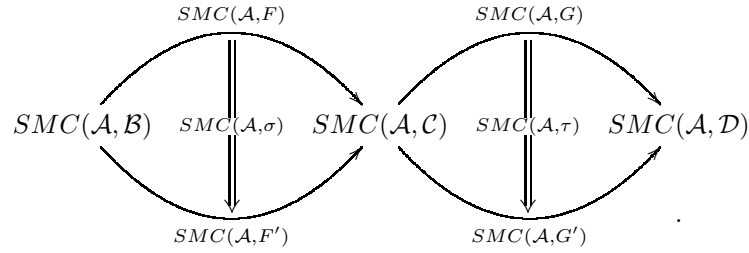
$$I \xrightarrow{G^0} G(I_{\mathcal{C}}) \xrightarrow{G(F^0)} GF(I_{\mathcal{B}})$$

in \mathcal{D} .

The last assertion results from the fact that SMC is a 2-category and therefore for any 2-cells $\sigma : F \rightarrow F'$ and $\tau : G \rightarrow G'$ in SMC , the image by $SMC(\mathcal{A}, -)$ of the composite 2-cell



is the composite of the 2-cells in **Cat**

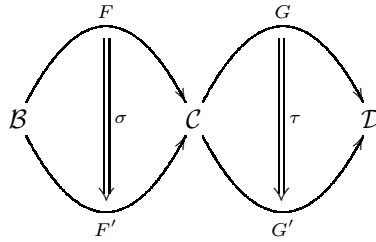


■

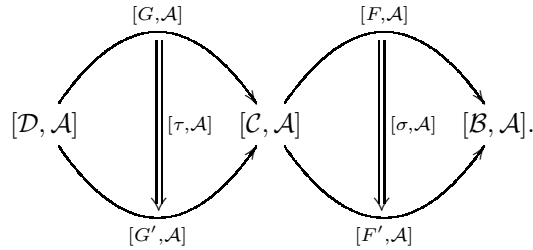
One has a similar lemma for the assignments $[-, \mathcal{A}]$.

Lemma 9.6 *The following statements hold in SMC for any \mathcal{A} .*

- For any \mathcal{B} , the 1-cell $[1_{\mathcal{B}}, \mathcal{A}]$ is the identity 1-cell at $[\mathcal{B}, \mathcal{A}]$.
- For any 1-cells $F : \mathcal{B} \rightarrow \mathcal{C}$ and $G : \mathcal{C} \rightarrow \mathcal{D}$, the 1-cells $[G \circ F, \mathcal{A}]$ and $[F, \mathcal{A}] \circ [G, \mathcal{A}] : [\mathcal{D}, \mathcal{A}] \rightarrow [\mathcal{B}, \mathcal{A}]$ are equal.
- For any 2-cells $\sigma : F \rightarrow F' : \mathcal{B} \rightarrow \mathcal{C}$ and $\tau : G \rightarrow G' : \mathcal{C} \rightarrow \mathcal{D}$, the image by $[-, \mathcal{A}]_{\mathcal{B}, \mathcal{D}}$ of the composite of the 2-cell



is the composite of the 2-cells



The proof is similar to the one for the previous lemma and even simpler since all the functors considered for the second assertion are strict.

Proposition 9.7 *One has a 2-functor $\underline{Hom} : SMC^{op} \times SMC \rightarrow SMC$ sending:*

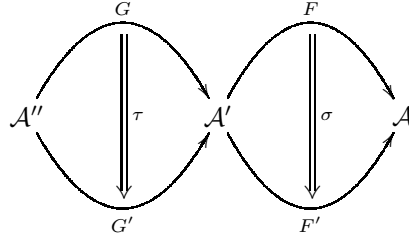
- *any pair $(\mathcal{A}, \mathcal{B})$ of symmetric monoidal categories to $[\mathcal{A}, \mathcal{B}]$;*
- *any pair of symmetric monoidal functors $(F : \mathcal{C} \rightarrow \mathcal{A}, G : \mathcal{B} \rightarrow \mathcal{D})$ to $[F, G] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{D}]$;*
- *any pair of 2-cells of SMC*

$$(\sigma : F \rightarrow F' : \mathcal{C} \rightarrow \mathcal{A}, \tau : G \rightarrow G' : \mathcal{B} \rightarrow \mathcal{D})$$

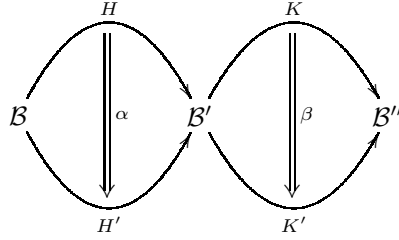
to $[\sigma, \tau] : [F, G] \rightarrow [F', G'] : [\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{C}, \mathcal{D}]$.

PROOF: One needs to check the following three assertions.

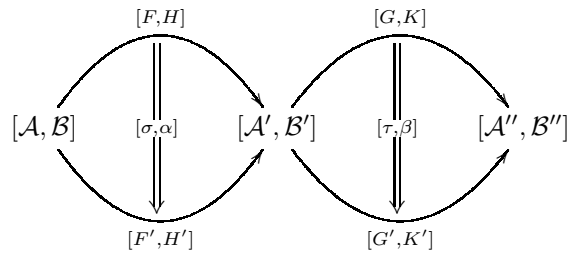
1. Given any $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} in SMC , the above assignments define a functor $SMC(\mathcal{C}, \mathcal{A}) \times SMC(\mathcal{B}, \mathcal{D}) \rightarrow SMC([\mathcal{A}, \mathcal{B}], [\mathcal{C}, \mathcal{D}])$.
2. Given any 2-cells in SMC



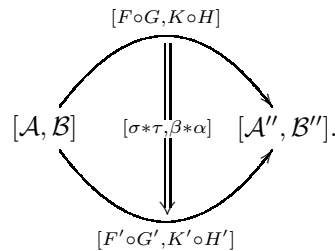
and



the composite 2-cell in SMC



is the 2-cell



3. Given any \mathcal{A} , and \mathcal{B} , the 1-cell $[1_{\mathcal{A}}, 1_{\mathcal{B}}]$ is the identity at $[\mathcal{A}, \mathcal{B}]$ in SMC .

Assertion 1. is Lemma 9.4.

2. is straightforward from Lemma 9.1, and the second and third assertions of Lemmas 9.5 and 9.6.

3. is straightforward from Lemma 9.1 and the first assertions of Lemmas 9.5 and 9.6. \blacksquare

We mention here the following that is a consequence the 2-functoriality of $\underline{Hom}(\mathcal{A}, -) : SMC \rightarrow SMC$ for any symmetric monoidal category \mathcal{A} and Lemma 2.12. (Transpose 2.12 to the case $\mathcal{V} = \mathbf{Cat}$ and $F = \underline{Hom}(\mathcal{A}, -) : SMC \rightarrow SMC$ for any symmetric monoidal category \mathcal{A} , to obtain the result.)

Corollary 9.8 *For any \mathcal{A} in SMC , the collection of mere functors*

$$Post_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : SMC(\mathcal{B}, \mathcal{C}) \rightarrow SMC([\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}])$$

for all \mathcal{B} and \mathcal{C} , defines a 2-natural transformation between 2-functors $SMC^{op} \times SMC \rightarrow \mathbf{Cat}$.

We can improve further this result.

Corollary 9.9 *For any \mathcal{A} and \mathcal{C} , the collection of 1-cells*

$$[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}} : [\mathcal{B}, \mathcal{C}] \rightarrow [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$$

in SMC , for all \mathcal{B} , defines a 2-natural transformation between functors

$$\underline{Hom}(-, \mathcal{C}) \rightarrow \underline{Hom}(-, [\mathcal{A}, \mathcal{C}]) \circ \underline{Hom}(\mathcal{A}, -) : SMC^{op} \rightarrow SMC.$$

PROOF: According to Corollary 9.8, we only need to check that for any symmetric monoidal $F : \mathcal{B}' \rightarrow \mathcal{B}$ the following diagram in SMC commutes

$$\begin{array}{ccc} [\mathcal{B}, \mathcal{C}] & \xrightarrow{[\mathcal{A}, -]} & [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]] \\ [F, \mathcal{C}] \downarrow & & \downarrow [[\mathcal{A}, F], 1] \\ [\mathcal{B}', \mathcal{C}] & \xrightarrow{[\mathcal{A}, -]} & [[\mathcal{A}, \mathcal{B}'], [\mathcal{A}, \mathcal{C}]], \end{array}$$

but this is immediate from Corollary 9.8 since all the monoidal functors involved here are strict. \blacksquare

Lemma 9.10 *For any \mathcal{A} and \mathcal{B} , the collection of 1-cells $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}} : [\mathcal{B}, \mathcal{C}] \rightarrow [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$ in SMC , for all \mathcal{C} , defines a 2-natural transformation between functors*

$$\underline{Hom}(\mathcal{B}, -) \rightarrow \underline{Hom}([\mathcal{A}, \mathcal{B}], -) \circ \underline{Hom}(\mathcal{A}, -) : SMC \rightarrow SMC.$$

PROOF: According to Corollary 9.8, it remains to show that for any symmetric monoidal functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, the following diagram in SMC commutes

$$\begin{array}{ccc} [\mathcal{B}, \mathcal{C}] & \xrightarrow{[\mathcal{A}, -]} & [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]] \\ [\mathcal{B}, F] \downarrow & & \downarrow [1, [\mathcal{A}, F]] \\ [\mathcal{B}, \mathcal{C}'] & \xrightarrow{[\mathcal{A}, -]} & [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}']] \end{array}$$

and for this, it is enough to check that the monoidal structures of the two legs above are the same.

The monoidal natural transformations considered below, namely:
 $([1, [\mathcal{A}, F]] \circ [\mathcal{A}, -])^0, ([\mathcal{A}, -] \circ [\mathcal{B}, F])^0, ([1, [\mathcal{A}, F]] \circ [\mathcal{A}, -])^2_{H, K}$ and, $([\mathcal{A}, -] \circ [\mathcal{B}, F])^2_{H, K},$

are between symmetric monoidal functors $[\mathcal{A}, \mathcal{B}] \rightarrow [\mathcal{A}, \mathcal{C}']$.

The monoidal natural transformation $([1, [\mathcal{A}, F]] \circ [\mathcal{A}, -])^0$ rewrites

$$I \xrightarrow{[1, [\mathcal{A}, F]]^0} [\mathcal{A}, F] \circ I \xlongequal{\quad} [\mathcal{A}, F] \circ [\mathcal{A}, I]$$

and has component in any $G : \mathcal{A} \rightarrow \mathcal{B}$, the arrow in $[\mathcal{A}, \mathcal{C}']$

$$I \xrightarrow{[\mathcal{A}, F]^0} F \circ I \xlongequal{\quad} F \circ I \circ G$$

which has component in any a in \mathcal{A} , the arrow $F^0 : I \rightarrow F(I)$ of \mathcal{C}' .

The monoidal natural transformation $([\mathcal{A}, -] \circ [\mathcal{B}, F])^0$ rewrites

$$I \xlongequal{\quad} [\mathcal{A}, I] \xrightarrow{[\mathcal{A}, [\mathcal{B}, F]^0]} [\mathcal{A}, F \circ I]$$

that has component in any $G : \mathcal{A} \rightarrow \mathcal{B}$, the arrow of $[\mathcal{A}, \mathcal{C}']$

$$I \xlongequal{\quad} I \circ G \xrightarrow{[\mathcal{B}, F]^0 * G} F \circ I \circ G$$

which has component in any a in \mathcal{A} , the arrow $F^0 : I \rightarrow F(I)$ of \mathcal{C}' .

Now consider any symmetric monoidal functors $H, K : \mathcal{B} \rightarrow \mathcal{C}$.

The monoidal natural transformation $([1, [\mathcal{A}, F]] \circ [\mathcal{A}, -])^2_{H, K}$ rewrites

$$([\mathcal{A}, F] \circ [\mathcal{A}, H]) \square ([\mathcal{A}, F] \circ [\mathcal{A}, K]) \xrightarrow{[1, [\mathcal{A}, F]]^2_{[\mathcal{A}, H], [\mathcal{A}, K]}} [\mathcal{A}, F] \circ ([\mathcal{A}, H] \square [\mathcal{A}, K]) \xlongequal{\quad} [\mathcal{A}, F] \circ [\mathcal{A}, H \square K]$$

and has component in any $G : \mathcal{A} \rightarrow \mathcal{B}$, the arrow of $[\mathcal{A}, \mathcal{C}']$

$$FHG \square FKG \xrightarrow{[\mathcal{A}, F]^2_{HG, KG}} F(HG \square KG) \xlongequal{\quad} F(H \square K)G$$

that has component in any a in \mathcal{A} , the arrow

$$F^2_{HGa, KGa} : FHGa \otimes FKGa \rightarrow F(HGa \otimes KGa)$$

of \mathcal{C}' .

The monoidal natural transformation $([\mathcal{A}, -] \circ [\mathcal{B}, F])^2_{H, K}$ rewrites

$$[\mathcal{A}, FH] \square [\mathcal{A}, FK] \xlongequal{\quad} [\mathcal{A}, FH \square FK] \xrightarrow{[\mathcal{A}, [\mathcal{B}, F]^2_{H, K}]} [\mathcal{A}, F(H \square K)]$$

that has component in any $G : \mathcal{A} \rightarrow \mathcal{B}$, the arrow of $[\mathcal{A}, \mathcal{C}']$

$$FHG \square FKG \xlongequal{\quad} (FH \square FK)G \xrightarrow{[\mathcal{B}, F]^2_{H, K} * G} F(H \square K)G$$

which has component in any a in \mathcal{A} , the arrow

$$F^2_{HGa, KGa} : FHGa \otimes FKGa \rightarrow F(HGa \otimes KGa)$$

of \mathcal{C}' . ■

Lemma 9.11 *The 2-cells in SMC*

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & [1, [G, 1]] & & \\
 & \swarrow & \parallel & \searrow & \\
 [\mathcal{A}, \mathcal{C}] & \xrightarrow{[\mathcal{B}, -]} & [[\mathcal{B}, \mathcal{A}], [\mathcal{B}, \mathcal{C}]] & & [[\mathcal{B}, \mathcal{A}], [\mathcal{B}', \mathcal{C}]] \\
 & \searrow & \parallel & \swarrow & \\
 & & [1, [\tau, 1]] & & \\
 & & \downarrow & & \\
 & & [1, [G', 1]] & &
 \end{array}
 \end{array}$$

and

$$\begin{array}{c}
 \begin{array}{ccccc}
 & & [[G, 1], 1] & & \\
 & \swarrow & \parallel & \searrow & \\
 [\mathcal{A}, \mathcal{C}] & \xrightarrow{[\mathcal{B}', -]} & [[\mathcal{B}', \mathcal{A}], [\mathcal{B}', \mathcal{C}]] & & [[\mathcal{B}, \mathcal{A}], [\mathcal{B}', \mathcal{C}]] \\
 & \searrow & \parallel & \swarrow & \\
 & & [[\tau, 1], 1] & & \\
 & & \downarrow & & \\
 & & [[G', 1], 1] & &
 \end{array}
 \end{array}$$

are equal, for any \mathcal{A} and \mathcal{C} and any 2-cell $\tau : G \rightarrow G' : \mathcal{B}' \rightarrow \mathcal{B}$.

PROOF: That the underlying 2-cells in **Cat** are equal, is equivalent to the fact that for any monoidal natural transformation $\sigma : F \rightarrow F' : \mathcal{A} \rightarrow \mathcal{C}$ the composite 2-cells in *SMC*

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & [\mathcal{B}, F] & \\
 \swarrow & \parallel & \searrow \\
 [\mathcal{B}, \mathcal{A}] & [\mathcal{B}, \sigma] & [\mathcal{B}, \mathcal{C}] \\
 \searrow & \parallel & \swarrow \\
 & [\mathcal{B}, F'] &
 \end{array}
 &
 &
 \begin{array}{ccc}
 & [G, \mathcal{C}] & \\
 \swarrow & \parallel & \searrow \\
 [\mathcal{B}, \mathcal{C}] & [\tau, \mathcal{C}] & [\mathcal{B}', \mathcal{C}] \\
 \searrow & \parallel & \swarrow \\
 & [G', \mathcal{C}] &
 \end{array}
 \end{array}$$

and

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & [G, \mathcal{A}] & \\
 \swarrow & \parallel & \searrow \\
 [\mathcal{B}, \mathcal{A}] & [\tau, \mathcal{A}] & [\mathcal{B}', \mathcal{A}] \\
 \searrow & \parallel & \swarrow \\
 & [G', \mathcal{A}] &
 \end{array}
 &
 &
 \begin{array}{ccc}
 & [\mathcal{B}', F] & \\
 \swarrow & \parallel & \searrow \\
 [\mathcal{B}', \mathcal{A}] & [\mathcal{B}', \sigma] & [\mathcal{B}', \mathcal{C}] \\
 \searrow & \parallel & \swarrow \\
 & [\mathcal{B}', F'] &
 \end{array}
 \end{array}$$

are equal, which is the case according to the 2-functoriality of Hom. To finish proving the lemma it remains to see that the diagram in *SMC*

$$\begin{array}{ccc}
 [\mathcal{A}, \mathcal{C}] & \xrightarrow{[\mathcal{B}, -]} & [[\mathcal{B}, \mathcal{A}], [\mathcal{B}, \mathcal{C}]] \\
 \downarrow [\mathcal{B}', -] & & \downarrow [1, [G, 1]] \\
 [[\mathcal{B}', \mathcal{A}], [\mathcal{B}, \mathcal{C}]] & \xrightarrow{[[G, 1], 1]} & [[\mathcal{B}, \mathcal{A}], [\mathcal{B}', \mathcal{C}]]
 \end{array}$$

commutes, but this is the case since all the symmetric monoidal functors considered here are strict.

■

10 Naturalities of the isomorphism D

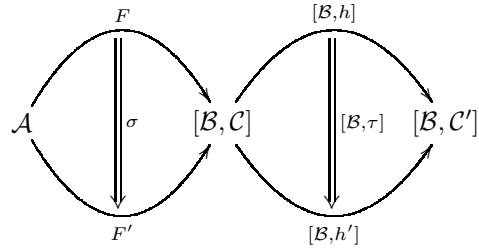
This section tackles the question of the naturalities of the isomorphism D .

Lemma 10.1 *The collection of isomorphisms defined in 6.2 defines a 2-natural transformation*

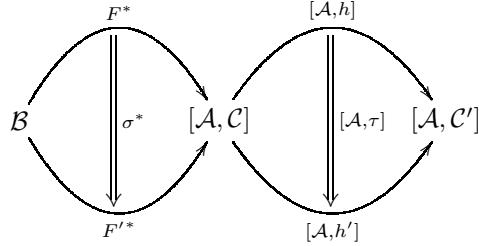
$$SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \cong_{\mathcal{A}, \mathcal{B}, \mathcal{C}} SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}])$$

between 2-functors $SMC^{op} \times SMC^{op} \times SMC \rightarrow \mathbf{Cat}$.

PROOF: To check the 2-naturality in \mathcal{C} , we show that given any 2-cells $\sigma : F \rightarrow F' : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ and $\tau : h \rightarrow h' : \mathcal{C} \rightarrow \mathcal{C}'$ in SMC , the 2-cells in SMC

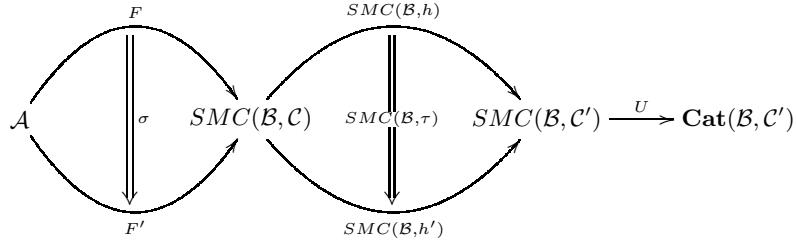


and

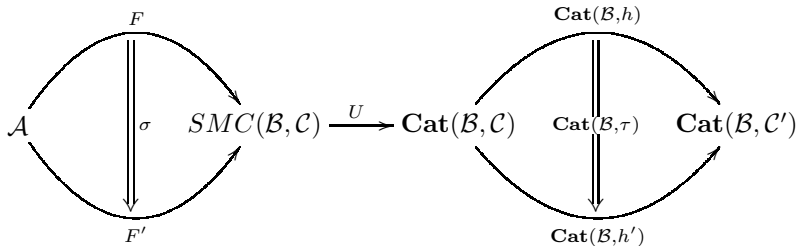


are equal.

Note that according to the 2-naturality in \mathcal{C} of $U : SMC(\mathcal{B}, \mathcal{C}) \rightarrow \mathbf{Cat}(\mathcal{B}, \mathcal{C})$, the mere natural transformation



is



which is dual in the sense of 6.1 to

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{F^*} & SMC(\mathcal{A}, \mathcal{C}) \\
 \downarrow \sigma^* & & \downarrow \\
 \mathcal{B} & \xrightarrow{F'^*} & SMC(\mathcal{A}, \mathcal{C})
 \end{array}
 \xrightarrow{U}
 \begin{array}{ccc}
 \mathbf{Cat}(\mathcal{A}, h) & \xrightarrow{\quad} & \mathbf{Cat}(\mathcal{A}, \mathcal{C}) \\
 \downarrow \mathbf{Cat}(\mathcal{A}, \tau) & & \downarrow \\
 \mathbf{Cat}(\mathcal{A}, h') & \xrightarrow{\quad} & \mathbf{Cat}(\mathcal{A}, \mathcal{C}')
 \end{array}$$

according to the 2-naturality of the isomorphism $\mathbf{Cat}(\mathcal{A}, \mathbf{Cat}(\mathcal{B}, \mathcal{C})) \cong \mathbf{Cat}(\mathcal{B}, \mathbf{Cat}(\mathcal{A}, \mathcal{C}))$ in \mathcal{C} . This last arrow is

$$\begin{array}{ccc}
 \mathcal{B} & \xrightarrow{F^*} & SMC(\mathcal{A}, \mathcal{C}) \\
 \downarrow \sigma^* & & \downarrow \\
 \mathcal{B} & \xrightarrow{F'^*} & SMC(\mathcal{A}, \mathcal{C})
 \end{array}
 \xrightarrow{SMC(\mathcal{A}, h)}
 \begin{array}{ccc}
 SMC(\mathcal{A}, \mathcal{C}) & \xrightarrow{SMC(\mathcal{A}, \tau)} & SMC(\mathcal{A}, \mathcal{C}') \\
 \downarrow SMC(\mathcal{A}, h') & & \downarrow \\
 SMC(\mathcal{A}, \mathcal{C}) & \xrightarrow{SMC(\mathcal{A}, \tau)} & SMC(\mathcal{A}, \mathcal{C}')
 \end{array}
 \xrightarrow{U} \mathbf{Cat}(\mathcal{A}, \mathcal{C}').$$

Therefore to prove the equality of the above 2-cells $[\mathcal{B}, \tau] * \sigma$ and $[\mathcal{A}, \tau] * \sigma^*$ of SMC , it remains to check that $[\mathcal{B}, h] \circ F$ and $[\mathcal{A}, h] \circ F^*$ are dual as symmetric monoidal functors and for this it is enough to show the four following points.

- (1) For any objects a, a' in \mathcal{A} and b in \mathcal{B} , the arrows $(([\mathcal{B}, h] \circ F)_{a, a'}^2)_b$ and $(([\mathcal{A}, h] \circ F^*)(b))_{a, a'}^2$ in \mathcal{C}' are equal.
- (2) For any objects b, b' in \mathcal{B} and a in \mathcal{A} , the arrows $(([\mathcal{A}, h] \circ F^*)_{b, b'}^2)_a$ and $(([\mathcal{B}, h] \circ F)(a))_{b, b'}^2$ in \mathcal{C}' are equal.
- (3) For any object b in \mathcal{B} , the arrows $([\mathcal{B}, h] \circ F)_b^0$ and $(([\mathcal{A}, h] \circ F^*)(b))^0$ in \mathcal{C}' are equal.
- (4) For any object a in \mathcal{A} , the arrows $([\mathcal{A}, h] \circ F^*)_a^0$ and $(([\mathcal{B}, h] \circ F)(a))^0$ in \mathcal{C}' are equal.

We only check below points (1) and (3). The computations for checking points (2) and (4) are the same respectively as for (1) and (3), up to adequate interchanges between F and F^* and between objects $a, a' \dots$ of \mathcal{A} and objects $b, b' \dots$ of \mathcal{B} .

To check (1), consider any objects a, a' in \mathcal{A} and b in \mathcal{B} . The arrow

$$([\mathcal{B}, h] \circ F)_{a, a'}^2 : ([\mathcal{B}, h] \circ F)(a) \square ([\mathcal{B}, h] \circ F)(a') \rightarrow ([\mathcal{B}, h] \circ F)(a \otimes a')$$

of $[\mathcal{B}, \mathcal{C}']$, is the monoidal natural transformation

$$(h \circ Fa) \square (h \circ Fa') \xrightarrow{[\mathcal{B}, h]_{Fa, Fa'}^2} h \circ (Fa \square Fa') \xrightarrow{h * F_{a, a'}^2} h \circ F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}'$$

which has component in b , the arrow

$$(i) : h(Fa(b)) \otimes h(Fa'(b)) \xrightarrow{h_{Fa(b), Fa'(b)}^2} h(Fa(b) \otimes Fa'(b)) \xrightarrow{h((F_{a, a'}^2)_b)} h(F(a \otimes a')(b))$$

in \mathcal{C}' . On the other hand $([\mathcal{A}, h] \circ F^*)(b) = h \circ F^*b$ and the arrow

$$(h \circ F^*b)_{a, a'}^2 : h(F^*b(a)) \otimes h(F^*b(a')) \rightarrow h(F^*b(a \otimes a'))$$

in \mathcal{C}' is

$$(ii) : h(F^*b(a)) \otimes h(F^*b(a')) \xrightarrow{h_{F^*b(a), F^*b(a')}^2} h(F^*b(a) \otimes F^*b(a')) \xrightarrow{h((F^*b)_{a,a'}^2)} h(F^*b(a \otimes a')).$$

Since $F^*b(a) = Fa(b)$ and $(F^*b)_{a,a'}^2 = (F_{a,a'}^2)_b$, arrows (i) and (ii) above are equal.

To check (3), consider an arbitrary object b in \mathcal{B} . The arrow $([\mathcal{B}, h] \circ F)^0$ of $[\mathcal{B}, \mathcal{C}']$ is the monoidal natural transformation

$$I \xrightarrow{[\mathcal{B}, h]^0} h \circ I_{[\mathcal{B}, \mathcal{C}]} \xrightarrow{h * F^0} h \circ (F(I_{\mathcal{A}})) : \mathcal{B} \rightarrow \mathcal{C}'$$

that has component in b the arrow

$$(i) : I_{\mathcal{C}'} \xrightarrow{h^0} h(I_{\mathcal{C}}) \xrightarrow{h(F_b^0)} h(F(I_{\mathcal{A}})(b))$$

in \mathcal{C}' . On the other hand, $([\mathcal{A}, h] \circ F^*)(b)$ is the arrow $(h \circ F^*b)^0 : I \rightarrow (h \circ F^*b)(I_{\mathcal{A}})$ in \mathcal{C}' , which is the composite

$$(ii) : I_{\mathcal{C}'} \xrightarrow{h^0} h(I_{\mathcal{C}}) \xrightarrow{h((F^*b)^0)} h(F^*b(I_{\mathcal{A}})).$$

Since $(F^*b)^0 = (F^0)_b$, arrows (i) and (ii) above are equal.

To prove the 2-naturality in \mathcal{B} , we show that given any 2-cells $\sigma : F \rightarrow F' : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ and $\tau : g \rightarrow g' : \mathcal{B}' \rightarrow \mathcal{B}$ in SMC , the 2-cells in SMC

and

are equal.

Note that the mere natural transformation

is

$$\begin{array}{c}
 \begin{array}{ccc}
 & F & \\
 \text{A} & \begin{array}{c} \downarrow \sigma \\ \downarrow \end{array} & SMC(\mathcal{B}, \mathcal{C}) \\
 & F' &
 \end{array}
 \xrightarrow{U}
 \begin{array}{ccc}
 & \mathbf{Cat}(g, \mathcal{C}) & \\
 \mathbf{Cat}(\mathcal{B}, \mathcal{C}) & \begin{array}{c} \downarrow \mathbf{Cat}(\tau, \mathcal{C}) \\ \downarrow \end{array} & \mathbf{Cat}(\mathcal{B}', \mathcal{C}) \\
 & \mathbf{Cat}(g', \mathcal{C}) &
 \end{array}
 \end{array}$$

which is dual in the sense of 6.1 to

$$\begin{array}{ccc}
 \begin{array}{ccc}
 & g & \\
 \mathcal{B}' & \begin{array}{c} \downarrow \tau \\ \downarrow \end{array} & \mathcal{B} \\
 & g' &
 \end{array}
 &
 \begin{array}{ccc}
 & F^* & \\
 SMC(\mathcal{A}, \mathcal{C}) & \begin{array}{c} \downarrow \sigma^* \\ \downarrow \end{array} & \mathbf{Cat}(\mathcal{A}, \mathcal{C}) \\
 & F'^* &
 \end{array}
 &
 \xrightarrow{U}
 \end{array}$$

according to the 2-naturality in \mathcal{B} of the isomorphism $\mathbf{Cat}(\mathcal{A}, \mathbf{Cat}(\mathcal{B}, \mathcal{C})) \cong \mathbf{Cat}(\mathcal{B}, \mathbf{Cat}(\mathcal{A}, \mathcal{C}))$.

Therefore to prove the equality of the above 2-cells $[\tau, \mathcal{C}] * \sigma$ and $\sigma^* * \tau$ of SMC , it remains to check that $[g, \mathcal{C}] \circ F$ and $F^* \circ g$ are dual as symmetric monoidal functors and for this it is enough to show the following four points.

- (1) For any objects a, a' in \mathcal{A} and x in \mathcal{B}' , the arrows $(([g, \mathcal{C}] \circ F)_{a, a'}^2)_x$ and $((F^* \circ g)(x))_{a, a'}^2$ in \mathcal{C} are equal.
- (2) For any objects a in \mathcal{A} and x, x' in \mathcal{B}' , the arrows $(([g, \mathcal{C}] \circ F)(a))_{x, x'}^2$ and $((F^* \circ g)_{x, x'}^2)_a$ in \mathcal{C} are equal.
- (3) For any object x in \mathcal{B}' , the arrows $([g, \mathcal{C}] \circ F)_x^0$ and $((F^* \circ g)(x))^0$ in \mathcal{C} are equal.
- (4) For any object a in \mathcal{A} , the arrows $(([g, \mathcal{C}] \circ F)(a))^0$ and $(F^* \circ g)_a^0$ in \mathcal{C} are equal.

To check point (1), consider any objects a, a' in \mathcal{A} , and x in \mathcal{B}' . The arrow $([g, \mathcal{C}] \circ F)_{a, a'}^2$ in $[\mathcal{B}', \mathcal{C}]$ is the monoidal natural transformation

$$(Fa \circ g) \square (Fa' \circ g) \Longrightarrow (Fa \square Fa') \circ g \xrightarrow{F_{a, a'}^2 * g} F(a \otimes a') \circ g : \mathcal{B}' \rightarrow \mathcal{C}$$

since $[g, \mathcal{C}]$ is strict, and it has component in x the arrow in \mathcal{C}

$$(F_{a, a'}^2)_{g(x)} : Fa(g(x)) \otimes Fa'(g(x)) \rightarrow F(a \otimes a')(g(x))$$

(i.e. the component of $F_{a, a'}^2 : Fa \square Fa' \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}$ in $g(x)$). This arrow is also $(F^*(g(x)))_{a, a'}^2$ since F and F^* are dual.

To check point (2), consider any objects a in \mathcal{A} , and x, x' in \mathcal{B}' . The arrow $(([g, \mathcal{C}] \circ F)(a))_{x, x'}^2$ of \mathcal{C} is $(Fa \circ g)_{x, x'}^2$ which is

$$(i) : Fa(g(x)) \otimes Fa(g(x')) \xrightarrow{(Fa)_{g(x), g(x')}}^2 Fa(g(x) \otimes g(x')) \xrightarrow{Fa(g_{x, x'}^2)} Fa(g(x \otimes x')).$$

On the other hand the arrow $(F^* \circ g)_{x, x'}^2$ in $[\mathcal{A}, \mathcal{C}]$ is the monoidal natural transformation

$$F^*(g(x)) \square F^*(g(x')) \xrightarrow{(F^*)_{g(x), g(x')}}^2 F^*(g(x) \otimes g(x')) \xrightarrow{F^*(g_{x, x'}^2)} F^*(g(x \otimes x')) : \mathcal{A} \rightarrow \mathcal{C}$$

and has component in a

$$(ii) : F^*(g(x))(a) \square F^*(g(x'))(a) \xrightarrow{((F^*)^2_{g(x), g(x')})_a} F^*(g(x) \otimes g(x'))(a) \xrightarrow{F^*(g^2_{x, x'})_a} F^*(g(x \otimes x'))(a).$$

Since F and F^* are dual, arrows (i) and (ii) above are equal.

Let us check point (3). The arrow $([g, \mathcal{C}] \circ F)^0$ in $[\mathcal{B}', \mathcal{C}]$ is the monoidal natural transformation

$$I \equiv I_{[\mathcal{B}, \mathcal{C}]} \circ g \xrightarrow{F^0 * g} F(I_{\mathcal{A}}) \circ g : \mathcal{B}' \rightarrow \mathcal{C}$$

since $[g, \mathcal{C}]$ is strict. For any object x in \mathcal{B}' , its component in x is the arrow in \mathcal{C}

$$(F^0)_{g(x)} : I \rightarrow F(I_{\mathcal{A}})(g(x))$$

that is also $(F^*(g(x)))^0$ since F and F^* are dual.

Let us check point (4). Consider any object a of \mathcal{A} . The arrow $(([g, \mathcal{C}] \circ F)(a))^0$ of \mathcal{C} is $(Fa \circ g)^0$ which is

$$(i) : I \xrightarrow{(Fa)^0} Fa(I_{\mathcal{B}}) \xrightarrow{Fa(g^0)} Fa(g(I_{\mathcal{B}'})).$$

On the other hand, the arrow $(F^* \circ g)^0$ of $[\mathcal{A}, \mathcal{C}]$ is the monoidal natural transformation

$$I \xrightarrow{(F^*)^0} F^*(I_{\mathcal{B}}) \xrightarrow{F^*(g^0)} F^*(g(I_{\mathcal{B}'})) : \mathcal{A} \rightarrow \mathcal{C}$$

and has component in a

$$(ii) : I_{[\mathcal{A}, \mathcal{C}]}(a) \xrightarrow{(F^*)^0_a} F^*(I_{\mathcal{B}})(a) \xrightarrow{F^*(g^0)_a} F^*(g(I_{\mathcal{B}'}))(a).$$

Since F and F^* are dual, the arrows (i) and (ii) above are equal.

The 2-naturality in \mathcal{A} results from the 2-naturality in \mathcal{B} since the isomorphisms 6.2 in **Cat**, $SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}])$ and $SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}]) \rightarrow SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$ are inverses in **Cat**. ■

Given any symmetric monoidal categories \mathcal{B} and \mathcal{C} , since D defines a 2-natural transformation in \mathcal{A} between 2-functors $SMC \rightarrow \mathbf{Cat}$

$$SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}]),$$

the Yoneda Lemma (in its enriched version) indicates that the collection of arrows D in **Cat** above is determined by a unique object in $SMC(\mathcal{B}, [[\mathcal{B}, \mathcal{C}], \mathcal{C}])$, namely the image by D of the identity in $[\mathcal{B}, \mathcal{C}]$, which we denote q . Precisely, one has the following.

Lemma 10.2 *For any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} , the diagram below between mere functors commute.*

$$\begin{array}{ccc} SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) & \xrightarrow{D} & SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}]) \\ & \searrow \underline{Hom}(-, \mathcal{C}) \quad \nearrow SMC(q, 1) & \\ & SMC([\mathcal{B}, \mathcal{C}], [\mathcal{C}], [\mathcal{A}, \mathcal{C}]) & \end{array}$$

Since the collection of arrows $D_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}])$ is also 2-natural in the arguments \mathcal{B} and \mathcal{C} , the collection of arrows $q : \mathcal{B} \rightarrow [[\mathcal{B}, \mathcal{C}], \mathcal{C}]$ is also 2-natural in the arguments \mathcal{B} and \mathcal{C} . We will use this point in the next chapter.

We can improve slightly Lemma 10.2.

Lemma 10.3 *The diagram in SMC*

$$\begin{array}{ccc} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{D} & [\mathcal{B}, [\mathcal{A}, \mathcal{C}]] \\ & \searrow [-, \mathcal{C}] \quad \nearrow [q, 1] & \\ & [[[\mathcal{B}, \mathcal{C}], \mathcal{C}], [\mathcal{A}, \mathcal{C}]] & \end{array}$$

commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} .

PROOF: We check that the composite $[q, 1] \circ [-, \mathcal{C}]$ is strict.

For any 1-cells $F, G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ in SMC , the arrow $([q, 1] \circ [-, \mathcal{C}])_{F, G}^2$ in $[\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$ is

$$([F, 1] \circ q) \square ([G, 1] \circ q) \xlongequal{\quad} ([F, 1] \square [G, 1]) \circ q \xrightarrow{[-, \mathcal{C}]_{F, G}^2 * q} [F \square G, 1] \circ q$$

since $[q, 1]$ is strict. The above arrow

$$[-, \mathcal{C}]_{F, G}^2 : [F, 1] \square [G, 1] \rightarrow [F \square G, 1]$$

in $[[[\mathcal{B}, \mathcal{C}], \mathcal{C}], [\mathcal{A}, \mathcal{C}]]$ is the natural transformation with component in any $H : [\mathcal{B}, \mathcal{C}] \rightarrow \mathcal{C}$, the arrow in $[\mathcal{A}, \mathcal{C}]$

$$(H \circ F) \square (H \circ G) \rightarrow H \circ (F \square G)$$

which is the natural transformation with component in any a of \mathcal{A} , the arrow

$$H_{Fa, Ga}^2 : HFa \otimes HGa \rightarrow H \circ (Fa \square Ga)$$

in \mathcal{C} . So that if the H above is strict, the component in H of $[-, \mathcal{C}]_{F, G}^2$ is an identity. Now for any b in \mathcal{B} , the component in b of the monoidal natural transformation $[-, \mathcal{C}]_{F, G}^2 * q$ is the component of $[-, \mathcal{C}]_{F, G}^2$ in the strict ev_b and thus it is an identity.

The arrow $([q, 1] \circ [-, \mathcal{C}])^0$ in $[\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$ is

$$I \xlongequal{\quad} I_{[[[\mathcal{B}, \mathcal{C}], \mathcal{C}], [\mathcal{A}, \mathcal{C}]]} \circ q \xrightarrow{[-, \mathcal{C}]^0 * q} [I_{[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]}, \mathcal{C}] \circ q.$$

The above arrow $[-, \mathcal{C}]^0 : I \rightarrow [I_{[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]}, \mathcal{C}]$ in $[[[\mathcal{B}, \mathcal{C}], \mathcal{C}], [\mathcal{A}, \mathcal{C}]]$ is the natural transformation with component in any $H : [\mathcal{B}, \mathcal{C}] \rightarrow \mathcal{C}$, the arrow $I \rightarrow H \circ I_{[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]}$ of $[\mathcal{A}, \mathcal{C}]$ which is the natural transformation with component in any a of \mathcal{A} the arrow $H^0 : I \rightarrow H(I_{[\mathcal{B}, \mathcal{C}]})$ in \mathcal{C} . So that if H as above is strict, the component in H of $[-, \mathcal{C}]^0$ is an identity. Now for any b in \mathcal{B} , the component in b of the monoidal natural transformation $[-, \mathcal{C}]^0 * q$ is the component of $[-, \mathcal{C}]^0$ in the strict ev_b and thus it is an identity.

■

Lemma 10.1 also implies in particular points (1),(2) and (3) below, which we will often use.

(1) Given any symmetric monoidal functors $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ and $G : \mathcal{C} \rightarrow \mathcal{C}'$, the following diagram in SMC is commutative

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F^*} & [\mathcal{A}, \mathcal{C}] \\ & \searrow ([\mathcal{B}, G] \circ F)^* & \downarrow [\mathcal{A}, G] \\ & & [\mathcal{A}, \mathcal{C}'] \end{array}$$

(2) Given any symmetric monoidal functors $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ and $G : \mathcal{B}' \rightarrow \mathcal{B}$, the following diagram in SMC is commutative

$$\begin{array}{ccc} \mathcal{B}' & \xrightarrow{G} & \mathcal{B} \\ & \searrow ([G, \mathcal{C}] \circ F)^* & \downarrow F^* \\ & & [\mathcal{A}, \mathcal{C}] \end{array}$$

(3) Given any symmetric monoidal functors $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ and $G : \mathcal{A}' \rightarrow \mathcal{A}$, the following diagram in SMC is commutative

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{F^*} & [\mathcal{A}, \mathcal{C}] \\ & \searrow (F \circ G)^* & \downarrow [G, \mathcal{C}] \\ & & [\mathcal{A}', \mathcal{C}] \end{array}$$

Proposition 10.4 *The collection of isomorphisms defined in 6.2 defines a 2-natural transformation between functors $SMC^{op} \times SMC^{op} \times SMC \rightarrow SMC$:*

$$[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \cong [\mathcal{B}, [\mathcal{A}, \mathcal{C}]].$$

PROOF: For the 2-naturality in \mathcal{C} , according to Lemma 10.1, one just needs to check that for any arrow $f : \mathcal{C} \rightarrow \mathcal{C}'$ the following diagram in SMC commutes

$$\begin{array}{ccc} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{\cong} & [\mathcal{B}, [\mathcal{A}, \mathcal{C}]] \\ [\mathcal{A}, [\mathcal{B}, f]] \downarrow & & \downarrow [\mathcal{B}, [\mathcal{A}, f]] \\ [\mathcal{A}, [\mathcal{B}, \mathcal{C}']] & \xrightarrow{\cong} & [\mathcal{B}, [\mathcal{A}, \mathcal{C}']] \end{array}$$

and for this, it just remains to show that the monoidal structures of $D \circ [\mathcal{A}, [\mathcal{B}, f]]$ and $[\mathcal{B}, [\mathcal{A}, f]] \circ D$ are the same.

Since the monoidal functor D is strict, for any symmetric monoidal functors $F, G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, the equality of the arrows $(D \circ [\mathcal{A}, [\mathcal{B}, f]])_{F, G}^2$ and $([\mathcal{B}, [\mathcal{A}, f]] \circ D)_{F, G}^2$ in $[\mathcal{B}, [\mathcal{A}, \mathcal{C}']]$ is equivalent to the fact that the monoidal natural transformation

$$[\mathcal{B}, [\mathcal{A}, f]]_{F^*, G^*}^2 : ([\mathcal{A}, f] \circ F^*) \square ([\mathcal{A}, f] \circ G^*) \rightarrow [\mathcal{A}, f] \circ (F^* \square G^*) : \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{C}']$$

is dual to

$$[\mathcal{A}, [\mathcal{B}, f]]_{F, G}^2 : ([\mathcal{B}, f] \circ F) \square ([\mathcal{B}, f] \circ G) \rightarrow [\mathcal{B}, f] \circ (F \square G) : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}'].$$

Consider any objects a in \mathcal{A} and b in \mathcal{B} . The component in a of $([\mathcal{A}, [\mathcal{B}, f]])_{F, G}^2$ is the monoidal natural transformation

$$[\mathcal{B}, f]_{Fa, Ga}^2 : (f \circ Fa) \square (f \circ Ga) \rightarrow f \circ (Fa \square Ga) : \mathcal{B} \rightarrow \mathcal{C}'$$

that has component in b , the arrow

$$f_{Fa(b),Ga(b)}^2 : f(Fa(b)) \otimes f(Ga(b)) \rightarrow f(Fa(b) \otimes Ga(b))$$

in \mathcal{C}' . The component in b of $[\mathcal{B}, [\mathcal{A}, f]]_{F^*, G^*}^2$ is the monoidal natural transformation

$$[\mathcal{A}, f]_{F^*b, G^*b}^2 : (f \circ F^*b) \square (f \circ G^*b) \rightarrow f \circ (F^*b \square G^*b) : \mathcal{A} \rightarrow \mathcal{C}'$$

that has component in a the arrow

$$f_{F^*b(a), G^*b(a)}^2 : f(F^*b(a)) \otimes f(G^*b(a)) \rightarrow f(F^*b(a) \otimes G^*b(a))$$

in \mathcal{C}' .

Since D is strict, the equality of the arrows $(D \circ [\mathcal{A}, [\mathcal{B}, f]])^0$ and $([\mathcal{B}, [\mathcal{A}, f]] \circ D)^0$ in $[\mathcal{B}, [\mathcal{A}, \mathcal{C}']]$ amounts to the fact the monoidal natural transformations

$$[\mathcal{B}, [\mathcal{A}, f]]^0 : I \rightarrow [\mathcal{A}, f] \circ I_{[\mathcal{B}, [\mathcal{A}, \mathcal{C}]]} : \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{C}']$$

and

$$[\mathcal{A}, [\mathcal{B}, f]]^0 : I \rightarrow [\mathcal{B}, f] \circ I_{[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]} : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}']$$

are dual. Consider any objects a in \mathcal{A} and b in \mathcal{B} . The component in a of $[\mathcal{A}, [\mathcal{B}, f]]^0$ is the monoidal natural transformation $[\mathcal{B}, f]^0 : I \rightarrow f \circ I_{[\mathcal{B}, \mathcal{C}]} : \mathcal{B} \rightarrow \mathcal{C}'$, with component in b the arrow $f^0 : I_{\mathcal{C}'} \rightarrow f(I_{\mathcal{C}})$. The component in b of $[\mathcal{B}, [\mathcal{A}, f]]^0$ is the monoidal natural transformation $[\mathcal{A}, f]^0 : I \rightarrow f \circ I_{[\mathcal{A}, \mathcal{C}]} : \mathcal{A} \rightarrow \mathcal{C}'$, with component in a the arrow $f^0 : I_{\mathcal{C}'} \rightarrow f(I_{\mathcal{C}})$.

The 2-naturality in \mathcal{A} of $D_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \cong [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]$ is actually trivial from Lemma 10.1. To prove it, one just needs to check that for any $f : \mathcal{A}' \rightarrow \mathcal{A}$, the following diagram in SMC commutes

$$\begin{array}{ccc} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{\cong} & [\mathcal{B}, [\mathcal{A}, \mathcal{C}]] \\ \downarrow [f, [\mathcal{B}, \mathcal{C}]] & & \downarrow [\mathcal{B}, [f, \mathcal{C}]] \\ [\mathcal{A}', [\mathcal{B}, \mathcal{C}]] & \xrightarrow{\cong} & [\mathcal{B}, [\mathcal{A}', \mathcal{C}]] \end{array}$$

and for this, it just remains to see that the monoidal structures of $D \circ [f, [\mathcal{B}, \mathcal{C}]]$ and $[\mathcal{B}, [f, \mathcal{C}]] \circ D$ are the same. This is the case since all the symmetric monoidal functors in the two composites above are strict.

The 2-naturality in \mathcal{B} of $D_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ results from its 2-naturality in \mathcal{A} since the isomorphisms $D_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ and $D_{\mathcal{B}, \mathcal{A}, \mathcal{C}}$ are inverses in SMC . ■

The naturalities of the isomorphisms D are further investigated and the end of this section contains results improving on Proposition 10.4.

Lemma 10.5 *The two arrows of SMC*

$$\mathcal{A} \xrightarrow{F} [\mathcal{B}, \mathcal{C}] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}], [\mathcal{D}, \mathcal{C}]]$$

and

$$[\mathcal{D}, \mathcal{B}] \xrightarrow{[\mathcal{D}, F^*]} [\mathcal{D}, [\mathcal{A}, \mathcal{C}]] \xrightarrow{D} [\mathcal{A}, [\mathcal{D}, \mathcal{C}]]$$

are dual (via 6.2) for any $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ and any \mathcal{D} .

PROOF: According to Lemma 10.3, the above arrow $D \circ [\mathcal{D}, F^*]$ rewrites

$$[\mathcal{D}, \mathcal{B}] \xrightarrow{[\mathcal{D}, F^*]} [\mathcal{D}, [\mathcal{A}, \mathcal{C}]] \xrightarrow{[-, \mathcal{C}]} [[[\mathcal{A}, \mathcal{C}], \mathcal{C}], [\mathcal{D}, \mathcal{C}]] \xrightarrow{[q, 1]} [\mathcal{A}, [\mathcal{D}, \mathcal{C}]]$$

which, according to 10.1, has dual

$$\mathcal{A} \xrightarrow{q} [[\mathcal{A}, \mathcal{C}], \mathcal{C}] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, [\mathcal{A}, \mathcal{C}]], [\mathcal{D}, \mathcal{C}]] \xrightarrow{[[1, F^*], 1]} [[\mathcal{D}, \mathcal{B}], [\mathcal{D}, \mathcal{C}]]$$

that rewrites

$$\begin{aligned} \mathcal{A} &\xrightarrow{q} [[\mathcal{A}, \mathcal{C}], \mathcal{C}] \xrightarrow{[F^*, 1]} [\mathcal{B}, \mathcal{C}] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}], [\mathcal{D}, \mathcal{C}]] \\ \mathcal{A} &\xrightarrow{F} [\mathcal{B}, \mathcal{C}] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}], [\mathcal{D}, \mathcal{C}]] \end{aligned}$$

according to Corollary 9.9 and Lemma 10.2. ■

Lemma 10.6 *For any \mathcal{A} , \mathcal{B} and \mathcal{C} , the arrow $[-, \mathcal{C}]_{\mathcal{A}, \mathcal{B}}$ in SMC is equal to the composite*

$$[\mathcal{A}, \mathcal{B}] \xrightarrow{[1, q]} [\mathcal{A}, [[\mathcal{B}, \mathcal{C}], \mathcal{C}]] \xrightarrow{D} [[\mathcal{B}, \mathcal{C}], [\mathcal{A}, \mathcal{C}]].$$

PROOF: $[-, \mathcal{C}]_{\mathcal{A}, \mathcal{B}}$ has dual $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}} : [\mathcal{B}, \mathcal{C}] \rightarrow [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$. Since $q : \mathcal{B} \rightarrow [[\mathcal{B}, \mathcal{C}], \mathcal{C}]$ is the dual of $1 : [\mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{B}, \mathcal{C}]$, according to Lemma 10.5, the composite $D \circ [1, q]$ has also dual $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}}$. ■

Lemma 10.7 *The following diagram in SMC is commutative*

$$\begin{array}{ccc} [\mathcal{D}, \mathcal{C}] & \xrightarrow{[-, \mathcal{B}]} & [[\mathcal{C}, \mathcal{B}], [\mathcal{D}, \mathcal{B}]] \\ \downarrow [1, F^*] & & \downarrow [F, 1] \\ [\mathcal{D}, [\mathcal{A}, \mathcal{B}]] & \xrightarrow{D} & [\mathcal{A}, [\mathcal{D}, \mathcal{B}]] \end{array}$$

for any $F : \mathcal{A} \rightarrow [\mathcal{C}, \mathcal{B}]$ and any \mathcal{D} .

PROOF: According to Lemma 10.5, the arrow

$$[\mathcal{D}, \mathcal{C}] \xrightarrow{[\mathcal{D}, F^*]} [\mathcal{D}, [\mathcal{A}, \mathcal{B}]] \xrightarrow{D} [\mathcal{A}, [\mathcal{D}, \mathcal{B}]]$$

has dual

$$\mathcal{A} \xrightarrow{F} [\mathcal{C}, \mathcal{B}] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{C}], [\mathcal{D}, \mathcal{B}]].$$

Since the dual of

$$[-, \mathcal{B}]_{\mathcal{D}, \mathcal{C}} : [\mathcal{D}, \mathcal{C}] \rightarrow [[\mathcal{C}, \mathcal{B}], [\mathcal{D}, \mathcal{B}]]$$

is

$$[\mathcal{D}, -]_{\mathcal{C}, \mathcal{B}} : [\mathcal{C}, \mathcal{B}] \rightarrow [[\mathcal{D}, \mathcal{C}], [\mathcal{D}, \mathcal{B}]],$$

according to Lemma 10.1, the dual of

$$[\mathcal{D}, \mathcal{C}] \xrightarrow{[-, \mathcal{B}]} [[\mathcal{C}, \mathcal{B}], [\mathcal{D}, \mathcal{B}]] \xrightarrow{[F, 1]} [\mathcal{A}, [\mathcal{D}, \mathcal{B}]]$$

is also

$$\mathcal{A} \xrightarrow{F} [\mathcal{C}, \mathcal{B}] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{C}], [\mathcal{D}, \mathcal{B}]].$$

■

Lemma 10.8 *The following diagram in SMC commutes*

$$\begin{array}{ccc}
& [\mathcal{B}, \mathcal{D}] & \\
\swarrow [\mathcal{A}, -] & & \searrow [\mathcal{C}, -] \\
[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{D}]] & & [[\mathcal{C}, \mathcal{B}], [\mathcal{C}, \mathcal{D}]] \\
\downarrow [F^*, 1] & & \downarrow [F, 1] \\
[\mathcal{C}, [\mathcal{A}, \mathcal{D}]] & \xrightarrow{D} & [\mathcal{A}, [\mathcal{C}, \mathcal{D}]]
\end{array}$$

for any $F : \mathcal{A} \rightarrow [\mathcal{C}, \mathcal{B}]$ and any \mathcal{D} .

PROOF: According to Lemma 10.1, for any symmetric monoidal transformation $\sigma : G \rightarrow H : \mathcal{B} \rightarrow \mathcal{D}$, the composite in SMC

$$\mathcal{C} \xrightarrow{F^*} [\mathcal{A}, \mathcal{B}] \xrightarrow{[\mathcal{A}, G]} [\mathcal{A}, \mathcal{D}]$$

is dual to

$$\mathcal{A} \xrightarrow{F} [\mathcal{C}, \mathcal{B}] \xrightarrow{[\mathcal{C}, G]} [\mathcal{C}, \mathcal{D}]$$

and the monoidal natural transformation

$$[\mathcal{A}, \sigma] * F^* : [\mathcal{A}, G] \circ F^* \rightarrow [\mathcal{A}, H] \circ F^*$$

is dual to

$$[\mathcal{C}, \sigma] * F : [\mathcal{C}, G] \circ F \rightarrow [\mathcal{C}, H] \circ F.$$

Therefore the diagram in **Cat** (between mere functors) is commutative

$$\begin{array}{ccc}
& SMC(\mathcal{B}, \mathcal{D}) & \\
\swarrow \underline{Hom}(\mathcal{A}, -) & & \searrow \underline{Hom}(\mathcal{C}, -) \\
SMC([\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{D}]) & & SMC([\mathcal{C}, \mathcal{B}], [\mathcal{C}, \mathcal{D}]) \\
\downarrow SMC(F^*, 1) & & \downarrow SMC(F, 1) \\
[\mathcal{C}, [\mathcal{A}, \mathcal{D}]] & \xrightarrow{D} & [\mathcal{A}, [\mathcal{C}, \mathcal{D}]].
\end{array}$$

Now since the symmetric monoidal functors $[\mathcal{A}, -]$, $[\mathcal{C}, -]$, $[F^*, 1]$, $[F, 1]$ and D are all strict, the result follows. ■

11 Evaluation functors

This section gathers technical lemmas involving the arrows q , defined in Section 10, and the internal hom.

Consider any symmetric monoidal categories \mathcal{A} , \mathcal{B} and any object a of \mathcal{A} . The image of a by $q : \mathcal{A} \rightarrow [[\mathcal{A}, \mathcal{B}], \mathcal{B}]$ will be written ev_a . According to Remark 6.7, since the identity symmetric monoidal functor at $[\mathcal{A}, \mathcal{B}]$ is strict, ev_a is strict. The functor ev_a sends any symmetric monoidal functor $F : \mathcal{A} \rightarrow \mathcal{B}$ to Fa and any monoidal natural transformation between symmetric functors $\sigma : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$ to its component $\sigma_a : Fa \rightarrow Ga$ in a .

The 2-naturality of the collection of arrows $q : \mathcal{A} \rightarrow [[\mathcal{A}, \mathcal{B}], \mathcal{B}]$ in the argument \mathcal{B} is equivalent to the Lemma below.

Lemma 11.1 *The following holds in SMC, for any object \mathcal{A} .*

- *For any $F : \mathcal{B} \rightarrow \mathcal{C}$ the diagram*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{q} & [[\mathcal{A}, \mathcal{B}], \mathcal{B}] \\ q \downarrow & & \downarrow [1, F] \\ [[\mathcal{A}, \mathcal{C}], \mathcal{C}] & \xrightarrow{[[1, F], 1]} & [[\mathcal{A}, \mathcal{B}], \mathcal{C}] \end{array}$$

commutes.

- *For any 2-cell $\sigma : F \rightarrow G : \mathcal{B} \rightarrow \mathcal{C}$, the 2-cells*

$$\begin{array}{ccccc} & & [1, F] & & \\ & \swarrow & \parallel & \searrow & \\ \mathcal{A} & \xrightarrow{q} & [[\mathcal{A}, \mathcal{B}], \mathcal{B}] & & [[\mathcal{A}, \mathcal{B}], \mathcal{C}] \\ & \searrow & \parallel & \swarrow & \\ & & [1, G] & & \end{array}$$

and

$$\begin{array}{ccccc} & & [[\mathcal{A}, F], 1] & & \\ & \swarrow & \parallel & \searrow & \\ \mathcal{A} & \xrightarrow{q} & [[\mathcal{A}, \mathcal{C}], \mathcal{C}] & & [[\mathcal{A}, \mathcal{B}], \mathcal{C}] \\ & \searrow & \parallel & \swarrow & \\ & & [[\mathcal{A}, G], 1] & & \end{array}$$

are equal.

Corollary 11.2 *For any object a of a symmetric monoidal category \mathcal{A} , the following holds in SMC.*

- *For any $F : \mathcal{B} \rightarrow \mathcal{C}$, the diagram*

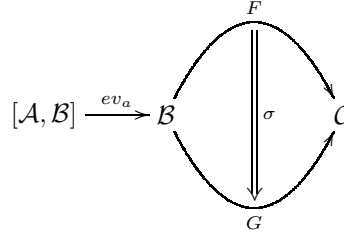
$$\begin{array}{ccc} [\mathcal{A}, \mathcal{B}] & \xrightarrow{ev_a} & \mathcal{B} \\ [\mathcal{A}, F] \downarrow & & \downarrow F \\ [\mathcal{A}, \mathcal{C}] & \xrightarrow{ev_a} & \mathcal{C} \end{array}$$

commutes.

- *For any 2-cell $\sigma : F \rightarrow G : \mathcal{B} \rightarrow \mathcal{C}$, the 2-cells*

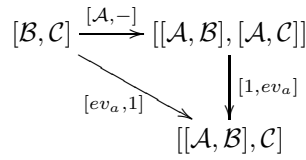
$$\begin{array}{ccccc} & & [\mathcal{A}, F] & & \\ & \swarrow & \parallel & \searrow & \\ [\mathcal{A}, \mathcal{B}] & & [\mathcal{A}, \sigma] & & [\mathcal{A}, \mathcal{C}] \xrightarrow{ev_a} \mathcal{C} \\ & \searrow & \parallel & \swarrow & \\ & & [\mathcal{A}, G] & & \end{array}$$

and



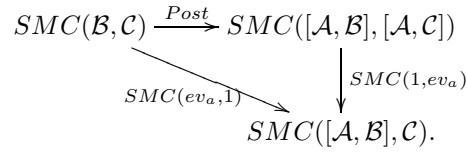
are equal.

Corollary 11.3 *The diagram in SMC*



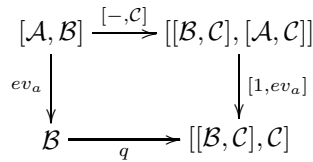
commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} , and any object a of \mathcal{A} .

PROOF: All the functors involved in the above diagram are strict, therefore it is enough to show that the diagram in **Cat** (between mere functors) is commutative



This is exactly Corollary 11.2. ■

Corollary 11.4 *The diagram in SMC*



commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} and any object a of \mathcal{A} .

PROOF: According to Lemma 10.1, $[1, ev_a] \circ [-, C]_{\mathcal{A}, \mathcal{B}}$ has dual

$$[\mathcal{B}, \mathcal{C}] \xrightarrow{[\mathcal{A}, -]} [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]] \xrightarrow{[1, ev_a]} [[\mathcal{A}, \mathcal{B}], \mathcal{C}]$$

since $[-, C]_{\mathcal{A}, \mathcal{B}}$ has dual $[\mathcal{A}, -]_{\mathcal{B}, \mathcal{C}}$, whereas $q \circ ev_a$ has dual

$$[\mathcal{B}, \mathcal{C}] \xrightarrow{[ev_a, 1]} [[\mathcal{A}, \mathcal{B}], \mathcal{C}]$$

since q has dual 1. So the result follows from Lemma 11.3. ■

The following result is immediate from Lemma 10.2.

Corollary 11.5 *The diagram in SMC*

$$\begin{array}{ccc}
 [\mathcal{A}, \mathcal{B}] & \xrightarrow{q} & [[[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]], [\mathcal{A}, \mathcal{C}] \\
 & \searrow [-, \mathcal{C}] & \swarrow [[\mathcal{A}, -], 1] \\
 & [[\mathcal{B}, \mathcal{C}], [\mathcal{A}, \mathcal{C}]]
 \end{array}$$

commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} .

Corollary 11.6 *The diagram in SMC*

$$\begin{array}{ccc}
 & [\mathcal{B}, \mathcal{C}] & \\
 & \downarrow [\mathcal{A}, -] & \searrow [F, \mathcal{C}] \\
 & [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]] & \\
 & \downarrow ev_F & \\
 & [\mathcal{A}, \mathcal{C}]. &
 \end{array}$$

commutes for any $F : \mathcal{A} \rightarrow \mathcal{B}$ and any \mathcal{C} .

Similarly one has from Lemma 10.2 the following.

Corollary 11.7 *The diagram in SMC*

$$\begin{array}{ccc}
 [\mathcal{A}, \mathcal{B}] & \xrightarrow{q} & [[[\mathcal{A}, \mathcal{B}], [\mathcal{C}, \mathcal{B}]], [\mathcal{C}, \mathcal{B}] \\
 & \searrow [\mathcal{C}, -] & \swarrow [[-, \mathcal{B}], 1] \\
 & [[\mathcal{C}, \mathcal{A}], [\mathcal{C}, \mathcal{B}]]
 \end{array}$$

commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} .

Corollary 11.8 *The diagram in SMC*

$$\begin{array}{ccc}
 & [\mathcal{C}, \mathcal{A}] & \\
 & \downarrow [-, \mathcal{B}] & \searrow [\mathcal{C}, F] \\
 & [[\mathcal{A}, \mathcal{B}], [\mathcal{C}, \mathcal{B}]] & \\
 & \downarrow ev_F & \\
 & [\mathcal{C}, \mathcal{B}] &
 \end{array}$$

commutes for any $F : \mathcal{A} \rightarrow \mathcal{B}$ and any \mathcal{C} .

Lemma 11.9 *The diagram in SMC*

$$\begin{array}{ccc}
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{D} & [\mathcal{B}, [\mathcal{A}, \mathcal{C}]] \\
 & \searrow ev_a & \swarrow [1, ev_a] \\
 & [\mathcal{B}, \mathcal{C}]
 \end{array}$$

commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} and any object a of \mathcal{A} .

PROOF: The above arrow $[1, ev_a] \circ D$ rewrites

1. $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \xrightarrow{[-, \mathcal{C}]} [[[\mathcal{B}, \mathcal{C}], \mathcal{C}], [\mathcal{A}, \mathcal{C}]] \xrightarrow{[q, 1]} [\mathcal{B}, [\mathcal{A}, \mathcal{C}]] \xrightarrow{[1, ev_a]} [\mathcal{B}, \mathcal{C}]$
2. $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \xrightarrow{[-, \mathcal{C}]} [[[\mathcal{B}, \mathcal{C}], \mathcal{C}], [\mathcal{A}, \mathcal{C}]] \xrightarrow{[1, ev_a]} [[[\mathcal{B}, \mathcal{C}], \mathcal{C}], \mathcal{C}] \xrightarrow{[q, 1]} [\mathcal{B}, \mathcal{C}]$
3. $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \xrightarrow{ev_a} [\mathcal{B}, \mathcal{C}] \xrightarrow{q} [[[\mathcal{B}, \mathcal{C}], \mathcal{C}], \mathcal{C}] \xrightarrow{[q, 1]} [\mathcal{B}, \mathcal{C}]$
4. $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \xrightarrow{ev_a} [\mathcal{B}, \mathcal{C}] \equiv [\mathcal{B}, \mathcal{C}].$

In the above derivation:

- arrow 1. is equal to $[1, ev_a] \circ D$ according to Lemma 10.3;
- arrows 2. and 3. are equal due to Corollary 11.4;
- arrows 3. and 4. are equal by Lemma 10.2 since q is dual to the identity. ■

12 The tensor $\mathcal{A} \otimes \mathcal{B}$ for symmetric monoidal categories \mathcal{A} and \mathcal{B}

This section is devoted to the definition by graph and relations of the symmetric monoidal category *tensor* $\mathcal{A} \otimes \mathcal{B}$ of any two symmetric monoidal categories \mathcal{A} and \mathcal{B} .

Let \mathcal{A} and \mathcal{B} stand for arbitrary symmetric monoidal categories.

We shall consider the following directed graph $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$, or simply \mathcal{H} in this section. Its set of vertices $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, or simply \mathcal{O} in this section, is the underlying set of the free algebra over $Obj(\mathcal{A}) \times Obj(\mathcal{B})$ for the signature consisting of the constant symbol I and the 2-ary symbol \otimes . Which is to say that \mathcal{O} is the set of words (or more accurately trees!) of the formal language composed according to the following axioms:

- I and all pairs (a, b) of $\mathcal{A} \times \mathcal{B}$, which we write $a \otimes b$, are in \mathcal{O} ;
- For any words X and Y in \mathcal{O} , the word $X \otimes Y$ is in \mathcal{O} .

The set of arrows of \mathcal{H} is also a formal language defined inductively. \mathcal{H} is the smallest graph on \mathcal{O} :

- containing the graphs $\mathcal{H}_{1, \mathcal{A}, \mathcal{B}}$, $\mathcal{H}_{2, \mathcal{A}, \mathcal{B}}$ and $\mathcal{H}_{3, \mathcal{A}, \mathcal{B}}$, just written respectively \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 in this section, all with set of vertices \mathcal{O} , and defined below;
- closed under the rules below of formation of new arrows.

The arrows of \mathcal{H}_1 are the “canonical arrows”, they are:

- for any X, Y, Z in \mathcal{O} , one $ass_{X, Y, Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ and one $\overline{ass}_{X, Y, Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$;
- for any X in \mathcal{O} , one $l_X : I \otimes X \rightarrow X$, one $\bar{l}_X : X \rightarrow I \otimes X$, one $r_X : X \otimes I \rightarrow X$ and one $\bar{r}_X : X \rightarrow X \otimes I$;
- for any X, Y in \mathcal{O} , one $s_{X, Y} : X \otimes Y \rightarrow Y \otimes X$.

\mathcal{H}_2 has the following set of arrows:

- for any object b of \mathcal{B} , one $\alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$;
- for any object a of \mathcal{A} , one $\beta_a : I \rightarrow a \otimes I_{\mathcal{B}}$;
- for any objects a, a' of \mathcal{A} and b of \mathcal{B} , one $\gamma_{a, a', b} : (a \otimes b) \otimes (a' \otimes b) \rightarrow (a \otimes a') \otimes b$;
- for any objects a of \mathcal{A} and b, b' of \mathcal{B} , one $\delta_{a, b, b'} : (a \otimes b) \otimes (a \otimes b') \rightarrow a \otimes (b \otimes b')$.

\mathcal{H}_3 consists of the following arrows:

- for any object a of \mathcal{A} and any arrow $f : b \rightarrow b'$ in \mathcal{B} , one $a \otimes f : a \otimes b \rightarrow a \otimes b'$;
- for any arrow $f : a \rightarrow a'$ of \mathcal{A} and any object b of \mathcal{B} , one $f \otimes b : a \otimes b \rightarrow a' \otimes b$.

For any X in \mathcal{O} and any arrow $p : Y \rightarrow Z$, there are new arrows in \mathcal{H} : $X \otimes p : X \otimes Y \rightarrow X \otimes Z$ and $p \otimes X : Y \otimes X \rightarrow Z \otimes X$.

The set of arrows of \mathcal{H}_1 , \mathcal{H}_2 and \mathcal{H}_3 are distinct and arrows in \mathcal{H} with distinct names are distinct.

Note here that since the categories \mathcal{A} and \mathcal{B} are small, $\mathcal{O}_{\mathcal{A},\mathcal{B}}$ is a “proper” set, and for any two X and Y in $\mathcal{O}_{\mathcal{A},\mathcal{B}}$, one has a “proper” set of arrows in $\mathcal{H}_{\mathcal{A},\mathcal{B}}$ from X to Y .

We need to consider the free category $\mathcal{F}_{\mathcal{A},\mathcal{B}}$, or simply \mathcal{F} in this section, generated by \mathcal{H} . Remember that its arrows are just concatenations of consecutive edges of \mathcal{H} . This is a small category.

For any X of \mathcal{O} , one has two graph endomorphisms of \mathcal{H} , namely $X \otimes -$ and $- \otimes X$, sending respectively any Y to $X \otimes Y$ (resp. $Y \otimes X$) and any arrow f to $X \otimes f$ (resp. $f \otimes X$). The graph endomorphisms $X \otimes -$ and $- \otimes X$ extend both to endofunctors of \mathcal{F} which are still denoted respectively by $X \otimes -$ and $- \otimes X$. Those are as follows:

- For any arrow $p_1 p_2 \dots p_n$ of \mathcal{F} where the p_i 's are consecutive edges in \mathcal{H} , $X \otimes p$ is the concatenation $(X \otimes p_1)(X \otimes p_2) \dots (X \otimes p_n)$, and $p \otimes X$ is $(p_1 \otimes X)(p_2 \otimes X) \dots (p_n \otimes X)$;
- For any X, Y in \mathcal{O} , $X \otimes 1_Y = 1_{X \otimes Y} = 1_X \otimes Y$, where the 1_X , 1_Y and $1_{X \otimes Y}$ stand for the identities respectively at X , Y and $X \otimes Y$ in \mathcal{F} (i.e some empty strings).

For any arrow f in \mathcal{A} and any object b in \mathcal{B} , we may write $f \otimes 1$ for the arrow $f \otimes b$, and for any object a in \mathcal{A} and any arrow g in \mathcal{B} , we may write $1 \otimes g$ for $a \otimes g$. Also for any arrow p of \mathcal{F} and any X in \mathcal{O} , we may write $1 \otimes p$ (respectively $p \otimes 1$) for the arrow $X \otimes p$, (resp. $p \otimes X$).

We introduce now relations on arrows of $\mathcal{F}_{\mathcal{A},\mathcal{B}}$ and define the *tensor* $\mathcal{A} \otimes \mathcal{B}$ of \mathcal{A} and \mathcal{B} as the category $\mathcal{F}_{\mathcal{A},\mathcal{B}} / \approx$, quotient of $\mathcal{F}_{\mathcal{A},\mathcal{B}}$ by the congruence \approx generated by these relations. These relations are the \sim indicated below from 12.1 to 12.23.

For all arrows $X \xrightarrow{t} Y$ and $Z \xrightarrow{s} W$ in \mathcal{H} ,

12.1

$$\begin{array}{ccccc}
 & & X \otimes W & & \\
 & \nearrow^{X \otimes s} & & \searrow^{t \otimes W} & \\
 X \otimes Z & & & & Y \otimes W. \\
 & \searrow_{t \otimes Z} & & \nearrow_{Y \otimes s} & \\
 & & Y \otimes Z & &
 \end{array}
 \quad \sim$$

12.2 Relations stating that canonical arrows of $\mathcal{A} \otimes \mathcal{B}$ are isomorphisms.

They are the following.

- For all X, Y, Z of \mathcal{O} , $X \otimes (Y \otimes Z) \xrightarrow{ass_{X,Y,Z}} (X \otimes Y) \otimes Z \xrightarrow{\overline{ass}_{X,Y,Z}} X \otimes (Y \otimes Z) \sim 1_{X \otimes (Y \otimes Z)}$

and $(X \otimes Y) \otimes Z \xrightarrow{ass_{X,Y,Z}} X \otimes (Y \otimes Z) \xrightarrow{\overline{ass}_{X,Y,Z}} (X \otimes Y) \otimes Z \sim 1_{(X \otimes Y) \otimes Z}$.

- For all X of \mathcal{O} , $I \otimes X \xrightarrow{l_X} X \xrightarrow{\bar{l}_X} I \otimes X \sim 1_{I \otimes X}$ and $X \xrightarrow{\bar{l}_X} I \otimes X \xrightarrow{l_X} X \sim 1_X$.

- For all X of \mathcal{O} , $X \otimes I \xrightarrow{r_X} X \xrightarrow{\bar{r}_X} X \otimes I \sim 1_{X \otimes I}$ and $X \xrightarrow{\bar{r}_X} X \otimes I \xrightarrow{r_X} X \sim 1_X$.

- For all X, Y of \mathcal{O} , $X \otimes Y \xrightarrow{s_{X,Y}} Y \otimes X \xrightarrow{s_{Y,X}} X \otimes Y \sim 1_{X \otimes Y}$.

12.3 Relations giving the coherence conditions for ass , r , l and s in $\mathcal{A} \otimes \mathcal{B}$.

These are the following.

- For all X, Y, Z and T in \mathcal{O} ,

$$\begin{array}{ccccc} X \otimes (Y \otimes (Z \otimes T)) & \xrightarrow{ass} & (X \otimes Y) \otimes (Z \otimes T) & \xrightarrow{ass} & ((X \otimes Y) \otimes Z) \otimes T \\ \downarrow 1 \otimes ass & & \sim & & \uparrow ass \otimes 1 \\ X \otimes ((Y \otimes Z) \otimes T) & \xrightarrow{ass} & & & (X \otimes (Y \otimes Z)) \otimes T. \end{array}$$

- For all X, Y in \mathcal{O} ,

$$\begin{array}{ccc} X \otimes (I \otimes Y) & \xrightarrow{ass} & (X \otimes I) \otimes Y \\ \searrow 1 \otimes l & \sim & \swarrow r \otimes 1 \\ & X \otimes Y. \end{array}$$

- For all X in \mathcal{O} ,

$$\begin{array}{ccc} X \otimes I & \xrightarrow{s} & I \otimes X \\ \searrow r & \sim & \swarrow l \\ & X. \end{array}$$

- For all X, Y and Z of \mathcal{O} ,

$$\begin{array}{ccccc} X \otimes (Y \otimes Z) & \xrightarrow{ass} & (X \otimes Y) \otimes Z & \xrightarrow{s} & Z \otimes (X \otimes Y) \\ \downarrow 1 \otimes s & & \sim & & \downarrow ass \\ X \otimes (Z \otimes Y) & \xrightarrow{ass} & (X \otimes Z) \otimes Y & \xleftarrow{s \otimes 1} & (Z \otimes X) \otimes Y. \end{array}$$

12.4 Relations for the naturalities of ass , r , l , and s in $\mathcal{A} \otimes \mathcal{B}$.

For instance, one has for any $f : X \rightarrow X'$ in \mathcal{H} , and any Y, Z in \mathcal{O} ,

$$\begin{array}{ccc} X \otimes (Y \otimes Z) & \xrightarrow{ass_{X,Y,Z}} & (X \otimes Y) \otimes Z \\ \downarrow f \otimes 1 & \sim & \downarrow (f \otimes 1) \otimes 1 \\ X' \otimes (Y \otimes Z) & \xrightarrow{ass_{X',Y,Z}} & (X' \otimes Y) \otimes Z. \end{array}$$

We will not write here the other relations. There are two more for the naturalities of $ass_{X,Y,Z}$ in Y and Z , one for that of l_X in X , one for that of r_X in X and two for those of $s_{X,Y}$ in X and Y .

For any object a in \mathcal{A} and any arrows $b \xrightarrow{f} b' \xrightarrow{g} b''$ in \mathcal{B} ,

12.5

$$\begin{array}{ccc} a \otimes b & \xrightarrow{a \otimes (g \circ f)} & a \otimes b'' \\ \searrow a \otimes f & \sim & \swarrow a \otimes g \\ & a \otimes b' \end{array}$$

For any object b in \mathcal{B} and any arrows $a \xrightarrow{f} a' \xrightarrow{g} a''$ in \mathcal{A} ,

12.6

$$\begin{array}{ccc} a \otimes b & \xrightarrow{(g \circ f) \otimes b} & a'' \otimes b \\ & \searrow f \otimes b \quad \sim \quad \nearrow g \otimes b & \\ & a' \otimes b & \end{array}$$

For any objects a in \mathcal{A} and b in \mathcal{B} ,

12.7 $a \otimes 1_b \sim 1_{a \otimes b}$

and

12.8 $1_a \otimes b \sim 1_{a \otimes b}$.

where 1_b , 1_a and $1_{a \otimes b}$ above are the identities respectively at b in \mathcal{B} , at a in \mathcal{A} and at $a \otimes b$ in \mathcal{F} .

For any arrows $f : a \rightarrow a'$ in \mathcal{A} and $g : b \rightarrow b'$ in \mathcal{B} ,

12.9

$$\begin{array}{ccc} a \otimes b & \xrightarrow{f \otimes b} & a' \otimes b \\ a \otimes g \downarrow & \sim & \downarrow a' \otimes g \\ a \otimes b' & \xrightarrow{f \otimes b'} & a' \otimes b'. \end{array}$$

12.10 Relations for the “naturalities” of α_b in b , β_a in a , $\gamma_{a,a',b}$ in a , a' and b and $\delta_{a,b,b'}$ in a , b and b' .

For instance by the relations for the “naturality” of $\gamma_{a,a',b}$ in b it is meant that for any objects a, a' in \mathcal{A} and any arrow $g : b \rightarrow b'$ in \mathcal{B} ,

$$\begin{array}{ccc} (a \otimes b) \otimes (a' \otimes b) & \xrightarrow{\gamma_{a,a',b}} & (a \otimes a') \otimes b \\ (1 \otimes g) \otimes (1 \otimes g) \downarrow & \sim & \downarrow 1 \otimes g \\ (a \otimes b') \otimes (a' \otimes b') & \xrightarrow{\gamma_{a,a',b'}} & (a \otimes a') \otimes b'. \end{array}$$

We will not write explicitly now the seven other relations.

For any objects a in \mathcal{A} and b, b', b'' in \mathcal{B} ,

12.11

$$\begin{array}{ccc} (a \otimes b) \otimes ((a \otimes b') \otimes (a \otimes b'')) & \xrightarrow{ass} & ((a \otimes b) \otimes (a \otimes b')) \otimes (a \otimes b'') \\ 1 \otimes \delta_{a,b',b''} \downarrow & & \downarrow \delta_{a,b,b'} \otimes 1 \\ (a \otimes b) \otimes (a \otimes (b' \otimes b'')) & \sim & (a \otimes (b \otimes b')) \otimes (a \otimes b'') \\ \delta_{a,b,b' \otimes b''} \downarrow & & \downarrow \delta_{a,b \otimes b',b''} \\ a \otimes (b \otimes (b' \otimes b'')) & \xrightarrow{1 \otimes ass_{b,b',b''}} & a \otimes ((b \otimes b') \otimes b''). \end{array}$$

For any objects a in \mathcal{A} and b in \mathcal{B} ,

12.12

$$\begin{array}{ccc}
 (a \otimes b) \otimes I & \xrightarrow{r_{a \otimes b}} & a \otimes b \\
 1 \otimes \beta_a \downarrow & \sim & \uparrow 1 \otimes r_b \\
 (a \otimes b) \otimes (a \otimes I_{\mathcal{B}}) & \xrightarrow{\delta_{a,b,I_{\mathcal{B}}}} & a \otimes (b \otimes I_{\mathcal{B}})
 \end{array}$$

and

12.13

$$\begin{array}{ccc}
 I \otimes (a \otimes b) & \xrightarrow{l_{a \otimes b}} & a \otimes b \\
 \beta_a \otimes 1 \downarrow & \sim & \uparrow l \otimes l_b \\
 (a \otimes I_{\mathcal{B}}) \otimes (a \otimes b) & \xrightarrow{\delta_{a,I_{\mathcal{B}},b}} & a \otimes (I_{\mathcal{B}} \otimes b).
 \end{array}$$

For any objects a in \mathcal{A} and b, b' in \mathcal{B} ,

12.14

$$\begin{array}{ccc}
 (a \otimes b) \otimes (a \otimes b') & \xrightarrow{\delta_{a,b,b'}} & a \otimes (b \otimes b') \\
 s_{a \otimes b, a \otimes b'} \downarrow & \sim & \downarrow 1 \otimes s_{b,b'} \\
 (a \otimes b') \otimes (a \otimes b) & \xrightarrow{\delta_{a,b',b}} & a \otimes (b' \otimes b).
 \end{array}$$

For any objects a, a', a'' in \mathcal{A} and b in \mathcal{B} ,

12.15

$$\begin{array}{ccc}
 (a \otimes b) \otimes ((a' \otimes b) \otimes (a'' \otimes b)) & \xrightarrow{ass} & ((a \otimes b) \otimes (a' \otimes b)) \otimes (a'' \otimes b) \\
 1 \otimes \gamma_{a',a'',b} \downarrow & & \downarrow \gamma_{a,a',b} \otimes 1 \\
 (a \otimes b) \otimes ((a' \otimes a'') \otimes b) & \sim & ((a \otimes a') \otimes b) \otimes (a'' \otimes b) \\
 \gamma_{a,a' \otimes a'',b} \downarrow & & \downarrow \gamma_{a \otimes a',a'',b} \\
 (a \otimes (a' \otimes a'')) \otimes b & \xrightarrow{ass \otimes 1} & ((a \otimes a') \otimes a'') \otimes b.
 \end{array}$$

For any objects a in \mathcal{A} and b in \mathcal{B} ,

12.16

$$\begin{array}{ccc}
 (a \otimes b) \otimes I & \xrightarrow{r_{a \otimes b}} & a \otimes b \\
 1 \otimes \alpha_b \downarrow & \sim & \uparrow r_a \otimes 1 \\
 (a \otimes b) \otimes (I_{\mathcal{A}} \otimes b) & \xrightarrow{\gamma_{a,I_{\mathcal{A}},b}} & (a \otimes I_{\mathcal{A}}) \otimes b
 \end{array}$$

and

12.17

$$\begin{array}{ccc}
 I \otimes (a \otimes b) & \xrightarrow{l_{a \otimes b}} & a \otimes b \\
 \alpha_b \otimes 1 \downarrow & \sim & \uparrow l_a \otimes 1 \\
 (I_{\mathcal{A}} \otimes b) \otimes (a \otimes b) & \xrightarrow{\gamma_{I_{\mathcal{A}},a,b}} & (I_{\mathcal{A}} \otimes a) \otimes b.
 \end{array}$$

For any objects a, a' in \mathcal{A} and b in \mathcal{B} ,

12.18

$$\begin{array}{ccc} (a \otimes b) \otimes (a' \otimes b) & \xrightarrow{\gamma_{a,a',b}} & (a \otimes a') \otimes b \\ s_{a \otimes b, a' \otimes b} \downarrow & \sim & \downarrow s_{a, a'} \otimes 1 \\ (a' \otimes b) \otimes (a \otimes b) & \xrightarrow{\gamma_{a', a, b}} & (a' \otimes a) \otimes b. \end{array}$$

For any objects b, b' in \mathcal{B} ,

12.19

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\cong} & I \\ \alpha_b \otimes \alpha_{b'} \downarrow & \sim & \downarrow \alpha_{b \otimes b'} \\ (I_{\mathcal{A}} \otimes b) \otimes (I_{\mathcal{A}} \otimes b') & \xrightarrow{\delta_{I_{\mathcal{A}}, b, b'}} & I_{\mathcal{A}} \otimes (b \otimes b'). \end{array}$$

For any objects a, a' in \mathcal{A} ,

12.20

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\cong} & I \\ \beta_a \otimes \beta_{a'} \downarrow & \sim & \downarrow \beta_{a \otimes a'} \\ (a \otimes I_{\mathcal{B}}) \otimes (a' \otimes I_{\mathcal{B}}) & \xrightarrow{\gamma_{a, a', I_{\mathcal{B}}}} & (a \otimes a') \otimes I_{\mathcal{B}}. \end{array}$$

12.21 $I \xrightarrow{\beta_{I_{\mathcal{A}}}} I_{\mathcal{A}} \otimes I_{\mathcal{B}} \sim I \xrightarrow{\alpha_{I_{\mathcal{B}}}} I_{\mathcal{A}} \otimes I_{\mathcal{B}}.$

For any objects a, a' in \mathcal{A} and b, b' in \mathcal{B} ,

12.22

$$\begin{array}{ccc} ((a \otimes b) \otimes (a \otimes b')) \otimes ((a' \otimes b) \otimes (a' \otimes b')) & \xrightarrow{\cong} & ((a \otimes b) \otimes (a' \otimes b)) \otimes ((a \otimes b') \otimes (a' \otimes b')) \\ \delta_{a, b, b'} \otimes \delta_{a', b, b'} \downarrow & \sim & \downarrow \gamma_{a, a', b} \otimes \gamma_{a, a', b'} \\ (a \otimes (b \otimes b')) \otimes (a' \otimes (b \otimes b')) & & ((a \otimes a') \otimes b) \otimes ((a \otimes a') \otimes b') \\ \gamma_{a, a', b \otimes b'} \downarrow & \nwarrow \delta_{a \otimes a', b, b'} & \\ (a \otimes a') \otimes (b \otimes b'). & & \end{array}$$

12.23 *Expansions of all relations above by iterations of $X \otimes -$ and $- \otimes X$ for all X in \mathcal{O} .*

Which means precisely that the set of relations \sim is the smallest set of relations on arrows of \mathcal{F} containing the previous relations (12.1 to 12.22) and satisfying the closure properties that for any relation $f \sim g : Y \rightarrow Z$ and any X in \mathcal{O} , one has the relations

$$X \otimes f \sim X \otimes g : X \otimes Y \rightarrow X \otimes Z$$

and

$$f \otimes X \sim g \otimes X : Y \otimes X \rightarrow Z \otimes X.$$

From now on, the arrows of $\mathcal{A} \otimes \mathcal{B}$, which are \approx -classes, will be denoted with the same name as their representatives in \mathcal{F} .

Now it is straightforward to check that the category $\mathcal{A} \otimes \mathcal{B}$ admits a symmetric monoidal structure. It is small since $\mathcal{F}_{\mathcal{A}, \mathcal{B}}$ is. According to the relations 12.23, the endofunctor $X \otimes -$ of \mathcal{F} sends equivalent arrows by \approx to equivalent arrows. Therefore there exists a unique endofunctor of the quotient category \mathcal{F}/\approx , still denoted $X \otimes -$, that makes the following diagram commute

$$\begin{array}{ccc} \mathcal{F} & \xrightarrow{X \otimes -} & \mathcal{F} \\ \downarrow & & \downarrow \\ (\mathcal{F}/\approx) & \xrightarrow{X \otimes -} & (\mathcal{F}/\approx) \end{array}$$

where the vertical arrows are the canonical ones associated to the quotient. One defines similarly from the endofunctor $- \otimes X$ of \mathcal{F} , the endofunctor $- \otimes X$ of \mathcal{F}/\approx .

From the relations 12.1, an induction shows that for any arrows $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ in \mathcal{F} , the arrows $X \otimes Y \xrightarrow{f \otimes Y} X' \otimes Y \xrightarrow{X' \otimes g} X' \otimes Y'$ and $X \otimes Y \xrightarrow{X \otimes g} X \otimes Y' \xrightarrow{f \otimes Y'} X' \otimes Y'$ are \approx -equivalent. Therefore for any arrows $f : X \rightarrow X'$ and $g : Y \rightarrow Y'$ of \mathcal{F}/\approx , $(f \otimes Y') \circ (X \otimes g) = (X' \otimes g) \circ (f \otimes Y)$ and the collections of endofunctors $X \otimes -$ and $- \otimes X$ for X ranging in \mathcal{O} , define a bifunctor $(\mathcal{F}/\approx) \times (\mathcal{F}/\approx) \rightarrow (\mathcal{F}/\approx)$.

That this bifunctor as tensor, together with the object I as unit and the *ass*, *r*, *l* and *s* as canonical arrows, define a symmetric monoidal structure, results from relations 12.2 (implying that arrows *ass*, *r*, *l* and *s* are isomorphisms and Axiom 2.3), relations 12.3 (Axioms 2.1, 2.2, 2.4 and 2.5) and relations 12.4 (naturalities of canonical arrows).

13 The embedding $\eta : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}]$

In this section a symmetric monoidal functor $\eta : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}]$ is defined for any symmetric monoidal categories \mathcal{A} and \mathcal{B} . It will serve to define the adjunction of the next section.

Let \mathcal{A} and \mathcal{B} stand here for arbitrary symmetric monoidal categories.

For any object a of \mathcal{A} , $\eta(a)$ denotes the following symmetric monoidal functor $\mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$. It sends any object b to $a \otimes b$ and any arrow $g : b \rightarrow b'$ to $a \otimes g : a \otimes b \rightarrow a \otimes b'$. These assignments define a functor $\mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$, according to the relations 12.5 and 12.7. The monoidal structure on $\eta(a)$ is defined by $\eta(a)^0 = \beta_a$ and for any objects b, b' in \mathcal{B} by $\eta(a)_{b, b'}^2 = \delta_{a, b, b'}$. The relations 12.10 for the naturalities of $\delta_{a, b, b'}$ in b and b' give the naturalities of $\eta(a)_{b, b'}^2$ in b and b' , and the relations 12.11, 12.12, 12.13 and 12.14 give respectively Axioms 2.6, 2.7, 2.8 and 2.9 for the triple $(\eta(a), \eta(a)^0, \eta(a)^2)$.

For any arrow $f : a \rightarrow a'$ in \mathcal{A} , the collection of arrows $f \otimes b : a \otimes b \rightarrow a' \otimes b$, b ranging in \mathcal{B} , forms a natural transformation $\eta(f) : \eta(a) \rightarrow \eta(a')$ according to the relations 12.9. This one is moreover monoidal with respect to the monoidal structures of $\eta(a)$ and $\eta(a')$: Axioms 2.10 and 2.11 for $\eta(f) : \eta(a) \rightarrow \eta(a')$ result respectively from the relations 12.10 for the naturality of the $\delta_{a, b, b'}$ in a and for the naturality of the β_a in a .

That the assignments $a \mapsto \eta(a)$ and $(f : a \rightarrow a') \mapsto \eta(f) : \eta(a) \rightarrow \eta(a')$ define a functor $\mathcal{A} \rightarrow \text{SMC}(\mathcal{B}, \mathcal{A} \otimes \mathcal{B})$ is due to the relations 12.6 and 12.8. We are going check that this functor admits the following symmetric monoidal structure $\mathcal{A} \rightarrow [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}]$. The monoidal natural transformation

$$\eta^0 : I \rightarrow \eta(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$$

is defined in any b of \mathcal{B} as $\alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ and for any objects a, a' in \mathcal{A} , the monoidal natural transformation

$$\eta_{a,a'}^2 : \eta(a) \square \eta(a') \rightarrow \eta(a \otimes a') : \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$$

is defined in any b of \mathcal{B} as

$$\gamma_{a,a',b} : (a \otimes b) \otimes (a' \otimes b) \rightarrow (a \otimes a') \otimes b.$$

The above collection η^0 is a well defined natural transformation $I \rightarrow \eta(I_{\mathcal{A}})$ according to the relations 12.10 for the naturality of the α_b in b . It is moreover monoidal with Axiom 2.10 given by the relations 12.19 and Axiom 2.11 by the relations 12.21.

For any objects a, a' of \mathcal{A} , the above collection $\eta_{a,a'}^2$ is a well defined natural transformation $\eta(a) \square \eta(a') \rightarrow \eta(a \otimes a')$ according to the relations 12.10 for the naturality of the $\gamma_{a,a',b}$ in b . It is moreover monoidal with Axiom 2.10 given by the relations 12.22, and Axiom 2.11 by the relations 12.20. The collection of $\eta_{a,a'}^2 : \eta(a) \square \eta(a') \rightarrow \eta(a \otimes a')$ in $[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}]$ is natural in a and a' , according to the relations 12.10 for the naturalities of the $\gamma_{a,a',b}$ in a and a' .

Eventually Axioms 2.6, 2.7, 2.8 and 2.9 hold for the triple (η, η^0, η^2) respectively due to the relations 12.15, 12.16, 12.17 and 12.18.

14 The symmetric monoidal adjunction $En \dashv Rn : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$

It is established in this section the existence of a symmetric monoidal adjunction

$$En \dashv Rn : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

such that $Rn \circ En = 1$, for any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} .

\mathcal{A} , \mathcal{B} and \mathcal{C} will stand here for arbitrary symmetric monoidal categories. The symmetric monoidal functor Rn is defined as the composite in SMC

$$[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \xrightarrow{[\mathcal{B}, -]} [[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}, \mathcal{C}]] \xrightarrow{[\eta, 1]} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]].$$

We embark now for the definition of the underlying functor

$$En : SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}).$$

On objects, it is as follows. Given any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, it is sent by En to the symmetric monoidal functor $\bar{F} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ described below.

The action of \bar{F} on $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$ is defined by induction according to the rules:

- $\bar{F}(I) = I$;
- For any objects a in \mathcal{A} and b in \mathcal{B} , $\bar{F}(a \otimes b) = F(a)(b)$;
- For any objects X, Y in $\mathcal{A} \otimes \mathcal{B}$, $\bar{F}(X \otimes Y) = \bar{F}(X) \otimes \bar{F}(Y)$.

To define the assignment \bar{F} on the arrows of $\mathcal{A} \otimes \mathcal{B}$, one first defines the graph morphism denoted \tilde{F} , from $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$ to the underlying graph of \mathcal{C} . For any X in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, $\tilde{F}(X) = \bar{F}(X)$.

\tilde{F} on arrows of $\mathcal{H}_{\mathcal{A}, \mathcal{B}}^1$ is as follows.

- For any X, Y, Z in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, \tilde{F} sends $a_{X, Y, Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ to

$$\bar{F}(X \otimes (Y \otimes Z)) = \bar{F}(X) \otimes (\bar{F}(Y) \otimes \bar{F}(Z)) \xrightarrow{ass_{\bar{F}(X), \bar{F}(Y), \bar{F}(Z)}} (\bar{F}(X) \otimes \bar{F}(Y)) \otimes \bar{F}(Z) = \bar{F}((X \otimes Y) \otimes Z)$$

and $\tilde{F}(a\bar{s}s_{X,Y,Z})$ is $ass_{\bar{F}(X),\bar{F}(Y),\bar{F}(Z)}^{-1}$.

- For any X in $\mathcal{O}_{\mathcal{A},\mathcal{B}}$, \tilde{F} sends $l_X : I \otimes X \rightarrow X$ to

$$\bar{F}(I \otimes X) \equiv I \otimes \bar{F}(X) \xrightarrow{l_{\bar{F}(X)}} \bar{F}(X)$$

and $\tilde{F}(\bar{l}_X)$ is $l_{\bar{F}(X)}^{-1}$; similarly $\tilde{F}(r_X)$ is $r_{\bar{F}(X)}$ and $\tilde{F}(\bar{r}_X)$ is $r_{\bar{F}(X)}^{-1}$.

- For any X, Y in $\mathcal{O}_{\mathcal{A},\mathcal{B}}$, \tilde{F} sends $s_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ to

$$\bar{F}(X \otimes Y) \equiv \bar{F}(X) \otimes \bar{F}(Y) \xrightarrow{s_{\bar{F}(X),\bar{F}(Y)}} \bar{F}(Y) \otimes \bar{F}(X) \equiv \bar{F}(Y \otimes X).$$

\tilde{F} on arrows of $\mathcal{H}_{\mathcal{A},\mathcal{B}}^2$ is as follows.

- For any object b of \mathcal{B} , \tilde{F} sends $\alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ to

$$\bar{F}(I) \equiv I \xrightarrow{F^0_b} F(I_{\mathcal{A}})(b) \equiv \bar{F}(I_{\mathcal{A}} \otimes b)$$

where F^0_b denotes the component in b of the monoidal natural transformation $F^0 : I \rightarrow F(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$, part of the monoidal structure of F .

- For any object a of \mathcal{A} , \tilde{F} sends $\beta_a : I \rightarrow a \otimes I_{\mathcal{B}}$ to

$$\bar{F}(I) \equiv I \xrightarrow{F(a)^0} F(a)(I_{\mathcal{B}}) \equiv \bar{F}(a \otimes I_{\mathcal{B}})$$

where $F(a)^0$ is part of the monoidal structure of $Fa : \mathcal{B} \rightarrow \mathcal{C}$.

- For any objects a, a' of \mathcal{A} and b of \mathcal{B} , \tilde{F} sends $\gamma_{a,a',b} : (a \otimes b) \otimes (a' \otimes b) \rightarrow (a \otimes a') \otimes b$ to

$$\bar{F}((a \otimes b) \otimes (a' \otimes b)) \equiv F(a)(b) \otimes F(a')(b) \xrightarrow{(F^2_{a,a'})_b} F(a \otimes a')(b) \equiv \bar{F}((a \otimes a') \otimes b)$$

where $(F^2_{a,a'})_b$ is the component in b of the monoidal transformation

$$F^2_{a,a'} : F(a) \square F(a') \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C},$$

part of the monoidal structure of F .

- For any objects a of \mathcal{A} and b, b' of \mathcal{B} , \tilde{F} sends $\delta_{a,b,b'} : (a \otimes b) \otimes (a \otimes b') \rightarrow a \otimes (b \otimes b')$ to

$$\bar{F}((a \otimes b) \otimes (a \otimes b')) \equiv F(a)(b) \otimes F(a)(b') \xrightarrow{F(a)^2_{b,b'}} F(a)(b \otimes b') \equiv \bar{F}(a \otimes (b \otimes b'))$$

where $F(a)^2_{b,b'}$ part of the monoidal structure of $F(a)$.

\tilde{F} on the arrows of $\mathcal{H}_{\mathcal{A},\mathcal{B}}^3$ is as follows.

- For any arrow $f : a \rightarrow a'$ in \mathcal{A} and any object b of \mathcal{B} , \tilde{F} sends $f \otimes b : a \otimes b \rightarrow a' \otimes b$ to $F(f)_b : F(a)(b) \rightarrow F(a')(b)$, the component in b of the monoidal natural transformation $F(f) : F(a) \rightarrow F(a')$.

- For any object a of \mathcal{A} and any arrow $g : b \rightarrow b'$ in \mathcal{B} , \tilde{F} sends $a \otimes g : a \otimes b \rightarrow a \otimes b'$ to $F(a)(g) : F(a)(b) \rightarrow F(a)(b')$.

\tilde{F} is defined on all arrows of $\mathcal{H}_{\mathcal{A},\mathcal{B}}$ by induction according to the following rules.

- For any object X of $\mathcal{O}_{\mathcal{A},\mathcal{B}}$ and any arrow $p : Y \rightarrow Z$ of $\mathcal{H}_{\mathcal{A},\mathcal{B}}$, \tilde{F} sends $X \otimes p : X \otimes Y \rightarrow X \otimes Z$ to

$$\bar{F}(X \otimes Y) \equiv \bar{F}(X) \otimes \bar{F}(Y) \xrightarrow{1 \otimes \bar{F}(p)} \bar{F}(X) \otimes \bar{F}(Z) \equiv \bar{F}(X \otimes Z)$$

and \tilde{F} sends $p \otimes X : Y \otimes X \rightarrow Z \otimes X$ to

$$\bar{F}(Y \otimes X) \equiv \bar{F}(Y) \otimes \bar{F}(X) \xrightarrow{\bar{F}(p) \otimes 1} \bar{F}(Z) \otimes \bar{F}(X) \equiv \bar{F}(Z \otimes X).$$

The above graph morphism \tilde{F} induces a functor $\mathcal{F}_{\mathcal{A}, \mathcal{B}} \rightarrow \mathcal{C}$, still written \tilde{F} . This last functor induces a functor $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$, namely \bar{F} , which we define as the image of F by En . To check this last point, it is sufficient to check that for any of the relations $f \sim g$ on arrows of $\mathcal{F}_{\mathcal{A}, \mathcal{B}}$ defining $\mathcal{A} \otimes \mathcal{B}$ (from 12.1 to 12.23), one has the equality $\tilde{F}(f) = \tilde{F}(g)$ in \mathcal{C} . Which we do below.

Regarding relations 12.1.

For any arrow $t : X \rightarrow Y$ and $s : Z \rightarrow W$ in $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$, \tilde{F} sends $X \otimes Z \xrightarrow{X \otimes s} X \otimes W \xrightarrow{t \otimes W} Y \otimes W$ to

$$\bar{F}(X) \otimes \bar{F}(Z) \xrightarrow{1 \otimes \tilde{F}(s)} \bar{F}(X) \otimes \bar{F}(Z) \xrightarrow{\tilde{F}(t) \otimes 1} \bar{F}(Y) \otimes \bar{F}(W)$$

and $X \otimes Z \xrightarrow{t \otimes Z} Y \otimes Z \xrightarrow{Y \otimes s} Y \otimes W$ to

$$\bar{F}(X) \otimes \bar{F}(Z) \xrightarrow{\tilde{F}(t) \otimes 1} \bar{F}(Y) \otimes \bar{F}(Z) \xrightarrow{1 \otimes \tilde{F}(s)} \bar{F}(Y) \otimes \bar{F}(W)$$

These two arrows are equal by the bifunctionality of the tensor in \mathcal{C} .

Relations 12.2, 12.3 and 12.4 are sent by \tilde{F} to commuting diagrams in \mathcal{C} . This results from the fact that \mathcal{C} is symmetric monoidal and that \tilde{F} sends:

- I to I ;
- objects of the form $X \otimes Y$ to $\bar{F}(X) \otimes \bar{F}(Y)$;
- arrows of the form $X \otimes f$ and $f \otimes X$ for any object X and any arrow f of $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$, respectively to $1 \otimes \tilde{F}(f)$ and to $\tilde{F}(f) \otimes 1$;
- the arrows ass to ass , the \overline{ass} to ass^{-1} , the l to l , the \bar{l} to l^{-1} , the r to r , the \bar{r} to r^{-1} and, the s to s .

Regarding relations 12.5 and 12.7.

For any object a of \mathcal{A} and any arrows $b \xrightarrow{f} b' \xrightarrow{g} b''$ in \mathcal{B} , \tilde{F} sends

$$(a \otimes b) \xrightarrow{a \otimes f} (a \otimes b') \xrightarrow{a \otimes g} (a \otimes b'')$$

to $F(a)(g) \circ F(a)(f)$, that is $F(a)(g \circ f)$ since $F(a) : \mathcal{B} \rightarrow \mathcal{C}$ is a functor, which is the image by \tilde{F} of $a \otimes (g \circ f) : a \otimes b \rightarrow a \otimes b''$.

For any objects a of \mathcal{A} and b of \mathcal{B} , \tilde{F} sends $a \otimes 1_b : a \otimes b \rightarrow a \otimes b$ to $F(a)(1_b)$, which is the identity at $F(a)(b)$ since $F(a)$ is a functor, which is the identity at $\bar{F}(a \otimes b)$ and the image by \tilde{F} of the identity at $a \otimes b$.

Regarding relations 12.6 and 12.8.

For any arrows $a \xrightarrow{f} a' \xrightarrow{g} a''$ in \mathcal{A} and any object b of \mathcal{B} , \tilde{F} sends

$$(a \otimes b) \xrightarrow{f \otimes b} (a' \otimes b) \xrightarrow{g \otimes b} (a'' \otimes b)$$

to $F(a)(b) \xrightarrow{F(f)_b} F(a')(b) \xrightarrow{F(g)_b} F(a'')(b)$, which is $F(g \circ f)_b : F(a)(b) \rightarrow F(a'')(b)$ by functoriality of F , which is the image by \tilde{F} of $(g \circ f) \otimes b : (a \otimes b) \rightarrow (a'' \otimes b)$.

For any objects a of \mathcal{A} and b of \mathcal{B} , the image of $1_a \otimes b : a \otimes b \rightarrow a \otimes b$ by \tilde{F} is the component in b of the natural transformation $F(1_a) : F(a) \rightarrow F(a)$, which is the identity at $F(a)(b)$ since F is a functor, which is the image by \tilde{F} of the identity at $a \otimes b$.

For any arrows $f : a \rightarrow a'$ in \mathcal{A} and $g : b \rightarrow b'$ in \mathcal{B} , the image by \tilde{F} of Diagram 12.9 is

$$\begin{array}{ccc} F(a)(b) & \xrightarrow{F(f)_b} & F(a')(b) \\ F(a)(g) \downarrow & & \downarrow F(a')(g) \\ F(a)(b') & \xrightarrow{F(f)_{b'}} & F(a')(b') \end{array}$$

which commutes by naturality of $F(f) : F(a) \rightarrow F(a') : \mathcal{B} \rightarrow \mathcal{C}$.

Regarding relations 12.10.

Naturality of α .

For any $f : b \rightarrow b'$ in \mathcal{B} , the diagram in $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$

$$\begin{array}{ccc} I & \xrightarrow{\alpha_b} & I_{\mathcal{A}} \otimes b \\ & \searrow \alpha_{b'} & \downarrow 1 \otimes f \\ & & I_{\mathcal{A}} \otimes b' \end{array}$$

is sent by \tilde{F} to

$$\begin{array}{ccc} I_{\mathcal{C}} & \xrightarrow{F_b^0} & F(I_{\mathcal{A}})(b) \\ & \searrow F_{b'}^0 & \downarrow F(I_{\mathcal{A}})(f) \\ & & F(I_{\mathcal{A}})(b') \end{array}$$

which commutes by naturality of $F^0 : I \rightarrow F(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$.

Naturality of β .

For any $f : a \rightarrow a'$ in \mathcal{A} , the diagram in $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$

$$\begin{array}{ccc} I & \xrightarrow{\beta_a} & a \otimes I_{\mathcal{B}} \\ & \searrow \beta_{a'} & \downarrow f \otimes 1 \\ & & a' \otimes I_{\mathcal{B}} \end{array}$$

is sent by \tilde{F} to

$$\begin{array}{ccc} I_{\mathcal{C}} & \xrightarrow{F(a)^0} & F(a)(I_{\mathcal{B}}) \\ & \searrow F(a')^0 & \downarrow F(f)_{I_{\mathcal{B}}} \\ & & F(a')(I_{\mathcal{B}}) \end{array}$$

which commutes according to Axiom 2.11 for the monoidal natural transformation $F(f) : F(a) \rightarrow F(a') : \mathcal{B} \rightarrow \mathcal{C}$.

Naturalities of γ .

For any $f : a \rightarrow c$ in \mathcal{A} and any objects a' of \mathcal{A} and b of \mathcal{B} , the diagram

$$\begin{array}{ccc} (a \otimes b) \otimes (a' \otimes b) & \xrightarrow{\gamma_{a,a',b}} & (a \otimes a') \otimes b \\ (f \otimes 1) \otimes 1 \downarrow & & \downarrow (f \otimes 1) \otimes 1 \\ (c \otimes b) \otimes (a' \otimes b) & \xrightarrow{\gamma_{c,a',b}} & (c \otimes a') \otimes b \end{array}$$

is sent by \tilde{F} to

$$\begin{array}{ccc} F(a)(b) \otimes F(a')(b) & \xrightarrow{(F_{a,a'}^2)_b} & F(a \otimes a')(b) \\ F(f)_b \otimes 1 \downarrow & & \downarrow F(f \otimes 1)_b \\ F(c)(b) \otimes F(a')(b) & \xrightarrow{(F_{c,a'}^2)_b} & F(c \otimes a')(b) \end{array}$$

which is pointwise in b the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc} F(a) \square F(a') & \xrightarrow{(F_{a,a'}^2)} & F(a \otimes a') \\ F(f) \square 1 \downarrow & & \downarrow F(f \otimes 1) \\ F(c) \square F(a') & \xrightarrow{(F_{c,a'}^2)} & F(c \otimes a') \end{array}$$

which commutes by naturality in a of the collection of $F_{a,a'}^2 : F(a) \square F(a') \rightarrow F(a \otimes a')$ in $[\mathcal{B}, \mathcal{C}]$.

Similarly the images by \tilde{F} of the diagram for the relations for the naturality of the $\gamma_{a,a',b}$ in a' are commutative diagrams according to the naturalities in a' of the collection of $F_{a,a'}^2$.

For any objects a, a' of \mathcal{A} and any arrow $g : b \rightarrow c$ in \mathcal{B} , the diagram

$$\begin{array}{ccc} (a \otimes b) \otimes (a' \otimes b) & \xrightarrow{\gamma_{a,a',b}} & (a \otimes a') \otimes b \\ (1 \otimes g) \otimes (1 \otimes g) \downarrow & & \downarrow (1 \otimes g) \\ (a \otimes c) \otimes (a' \otimes c) & \xrightarrow{\gamma_{a,a',c}} & (a \otimes a') \otimes c \end{array}$$

is sent by \tilde{F} to

$$\begin{array}{ccc} F(a)(b) \otimes F(a')(b) & \xrightarrow{(F_{a,a'}^2)_b} & F(a \otimes a')(b) \\ F(a)(g) \otimes F(a')(g) \downarrow & & \downarrow F(a \otimes a')(g) \\ F(a)(c) \otimes F(a')(c) & \xrightarrow{(F_{a,a'}^2)_c} & F(a \otimes a')(c) \end{array}$$

which commutes since $F_{a,a'}^2$ is a natural transformation $F_{a,a'}^2 : F(a) \square F(a') \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}$.

Naturalities of δ .

For any arrow $f : a \rightarrow c$ in \mathcal{A} and any objects b, b' in \mathcal{B} , the diagram

$$\begin{array}{ccc} (a \otimes b) \otimes (a \otimes b') & \xrightarrow{\delta_{a,b,b'}} & a \otimes (b \otimes b') \\ (f \otimes 1) \otimes (f \otimes 1) \downarrow & & \downarrow f \otimes 1 \\ (c \otimes b) \otimes (c \otimes b') & \xrightarrow{\delta_{c,b,b'}} & c \otimes (b \otimes b') \end{array}$$

is sent by \tilde{F} to

$$\begin{array}{ccc} F(a)(b) \otimes F(a)(b') & \xrightarrow{F(a)_{b,b'}^2} & F(a)(b \otimes b') \\ F(f)_b \otimes F(f)_{b'} \downarrow & & \downarrow F(f)_{b \otimes b'} \\ F(c)(b) \otimes F(c)(b') & \xrightarrow{F(c)_{b,b'}^2} & F(c)(b \otimes b') \end{array}$$

which commutes according to Axiom 2.10 for the monoidal natural transformation $F(f) : F(a) \rightarrow F(c) : \mathcal{B} \rightarrow \mathcal{C}$.

For any objects a in \mathcal{A} and b' in \mathcal{B} and any arrow $g : b \rightarrow c$ in \mathcal{B} , the diagram

$$\begin{array}{ccc} (a \otimes b) \otimes (a \otimes b') & \xrightarrow{\delta_{a,b,b'}} & a \otimes (b \otimes b') \\ (1 \otimes g) \otimes 1 \downarrow & & \downarrow 1 \otimes (g \otimes 1) \\ (a \otimes c) \otimes (a \otimes b') & \xrightarrow{\delta_{a,c,b'}} & a \otimes (c \otimes b') \end{array}$$

is sent by \tilde{F} to

$$\begin{array}{ccc} F(a)(b) \otimes F(a)(b') & \xrightarrow{F(a)_{b,b'}^2} & F(a)(b \otimes b') \\ F(a)(g) \otimes 1 \downarrow & & \downarrow F(a)(g \otimes 1) \\ F(a)(c) \otimes F(a)(b') & \xrightarrow{F(a)_{c,b'}^2} & F(a)(c \otimes b') \end{array}$$

which commutes according to the naturality of the collection of $F(a)_{b,b'}^2$ in b since $F(a)$ is monoidal.

Similarly, the images by \tilde{F} of diagrams for the relations for the naturalities of the $\delta_{a,b,b'}$ in b' are commutative diagrams since $F(a)$ is monoidal.

For any objects a in \mathcal{A} and b, b', b'' in \mathcal{B} , the image of Diagram 12.11 by \tilde{F} is

$$\begin{array}{ccc} F(a)(b) \otimes (F(a)(b') \otimes F(a)(b'')) & \xrightarrow{ass} & (F(a)(b) \otimes F(a)(b')) \otimes F(a)(b'') \\ 1 \otimes F(a)_{b',b''}^2 \downarrow & & \downarrow F(a)_{b,b'}^2 \otimes 1 \\ F(a)(b) \otimes F(a)(b' \otimes b'') & & F(a)(b \otimes b') \otimes F(a)(b'') \\ F(a)_{b,b' \otimes b''}^2 \downarrow & & \downarrow F(a)_{b \otimes b',b''}^2 \\ F(a)(b \otimes (b' \otimes b'')) & \xrightarrow{F(a)(ass_{b,b',b''})} & F(a)((b \otimes b') \otimes b'') \end{array}$$

which commutes according to Axiom 2.6 for the monoidal functor $F(a)$.

For any objects a in \mathcal{A} and b in \mathcal{B} , the image by \tilde{F} of Diagram 12.12 is

$$\begin{array}{ccc} F(a)(b) \otimes I & \xrightarrow{r_{F(a)(b)}} & F(a)(b) \\ 1 \otimes F(a)^0 \downarrow & & \uparrow F(a)(r_b) \\ F(a)(b) \otimes F(a)(I_{\mathcal{B}}) & \xrightarrow{F(a)^2_{b, I_{\mathcal{B}}}} & F(a)(b \otimes I_{\mathcal{B}}) \end{array}$$

which commutes according to Axiom 2.7 for the monoidal functor $F(a)$.

Similarly for any objects a in \mathcal{A} and b in \mathcal{B} , the image by \tilde{F} of Diagram 12.13 is a commutative diagram according to Axiom 2.8 for the monoidal functor $F(a)$.

Given any objects a in \mathcal{A} and b, b' in \mathcal{B} , the image by \tilde{F} of the Diagram 12.14 is

$$\begin{array}{ccc} F(a)(b) \otimes F(a)(b') & \xrightarrow{F(a)^2_{b, b'}} & F(a)(b \otimes b') \\ s_{F(a)(b), F(a)(b')} \downarrow & & \downarrow F(a)(s_{b, b'}) \\ F(a)(b') \otimes F(a)(b) & \xrightarrow{F(a)^2_{b', b}} & F(a)(b' \otimes b) \end{array}$$

which commutes according to Axiom 2.9 for the symmetric functor $F(a)$.

For any objects a, a', a'' of \mathcal{A} and b of \mathcal{B} , the image by \tilde{F} of Diagram 12.15 is

$$\begin{array}{ccc} F(a)(b) \otimes ((F(a')(b) \otimes F(a'')(b))) & \xrightarrow{ass} & (F(a)(b) \otimes F(a')(b)) \otimes F(a'')(b) \\ 1 \otimes (F^2_{a', a''})_b \downarrow & & \downarrow (F^2_{a, a'})_b \otimes 1 \\ F(a)(b) \otimes F(a' \otimes a'')(b) & & F(a \otimes a')(b) \otimes F(a'')(b) \\ (F^2_{a, a' \otimes a''})_b \downarrow & & \downarrow (F^2_{a \otimes a', a''})_b \\ F(a \otimes (a' \otimes a''))(b) & \xrightarrow{F(ass_{a, a', a''})_b} & F((a \otimes a') \otimes a'')(b) \end{array}$$

which is the pointwise version of the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc} F(a) \square ((F(a') \square F(a''))) & \xrightarrow{ass_{F(a), F(a'), F(a'')}} & (F(a) \square F(a')) \square F(a'') \\ 1 \square F^2_{a', a''} \downarrow & & \downarrow F^2_{a, a'} \square 1 \\ F(a) \square F(a' \otimes a'') & & F(a \otimes a') \square F(a'') \\ F^2_{a, a' \otimes a''} \downarrow & & \downarrow F^2_{a \otimes a', a''} \\ F(a \otimes (a' \otimes a'')) & \xrightarrow{F(ass_{a, a', a''})} & F((a \otimes a') \otimes a'') \end{array}$$

which commutes according to Axiom 2.6 for the monoidal functor F .

For any objects a of \mathcal{A} and b of \mathcal{B} , the image by \tilde{F} of Diagram 12.16 is

$$\begin{array}{ccc} F(a)(b) \otimes I & \xrightarrow{r_{F(a)(b)}} & F(a)(b) \\ 1 \otimes F_b^0 \downarrow & & \uparrow (F(r_a))_b \\ F(a)(b) \otimes F(I_{\mathcal{A}})(b) & \xrightarrow{(F^2_{a, I_{\mathcal{A}}})_b} & F(a \otimes I_{\mathcal{A}})(b) \end{array}$$

which is the pointwise version of the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc} F(a) \square I & \xrightarrow{\tau_{F(a)}} & F(a) \\ 1 \square F^0 \downarrow & & \uparrow F(r_a) \\ F(a) \square F(I_{\mathcal{A}}) & \xrightarrow{F_{a, I_{\mathcal{A}}}^2} & F(a \otimes I_{\mathcal{A}}) \end{array}$$

which is commutative according to Axiom 2.7 for the monoidal functor F .

Similarly for any objects a in \mathcal{A} and b in \mathcal{B} , the image by \tilde{F} of Diagram 12.17 commutes according to Axiom 2.8 for the monoidal functor F .

For any objects a, a' of \mathcal{A} and any b of \mathcal{B} , the image by \tilde{F} of Diagram 12.18 is

$$\begin{array}{ccc} F(a)(b) \otimes F(a')(b) & \xrightarrow{(F_{a, a'}^2)_b} & F(a \otimes a')(b) \\ \downarrow s_{F(a)(b), F(a')(b')} & & \downarrow F(s_{a, a'})_b \\ F(a')(b) \otimes F(a)(b) & \xrightarrow{(F_{a', a}^2)_b} & F(a' \otimes a)(b) \end{array}$$

which is the pointwise version of the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc} F(a) \square F(a') & \xrightarrow{F_{a, a'}^2} & F(a \otimes a') \\ \downarrow s_{F(a), F(a')} & & \downarrow F(s_{a, a'}) \\ F(a') \square F(a) & \xrightarrow{F_{a', a}^2} & F(a' \otimes a) \end{array}$$

that commutes according to Axiom 2.9 for the symmetric monoidal functor F .

For any objects b, b' in \mathcal{B} , the image of Diagram 12.19 by \tilde{F} is

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\cong} & I \\ F^0_b \otimes F^0_{b'} \downarrow & & \downarrow F^0_{b \otimes b'} \\ F(I_{\mathcal{A}})(b) \otimes F(I_{\mathcal{A}})(b') & \xrightarrow{(F(I_{\mathcal{A}}))^2_{b, b'}} & F(I_{\mathcal{A}})(b \otimes b') \end{array}$$

which commutes according to Axiom 2.10 for the monoidal natural transformation $F^0 : I \rightarrow F(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$.

For any objects a, a' in \mathcal{A} , the image of Diagram 12.20 by \tilde{F} is

$$\begin{array}{ccc} I \otimes I & \xrightarrow{\cong} & I \\ (F(a))^0 \otimes (F(a'))^0 \downarrow & & \downarrow (F(a \otimes a'))^0 \\ F(a)(I_{\mathcal{B}}) \otimes F(a')(I_{\mathcal{B}}) & \xrightarrow{(F_{a, a'}^2)_{I_{\mathcal{B}}}} & F(a \otimes a')(I_{\mathcal{B}}) \end{array}$$

which commutes according to Axiom 2.11 for the monoidal natural transformation

$$F_{a, a'}^2 : F(a) \square F(a') \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}.$$

Regarding relations 12.21. The image by \tilde{F} of $\beta_{I_A} : I \rightarrow I_A \otimes I_B$ is $(F(I_A))^0 : I \rightarrow F(I_A)(I_B)$ whereas the image by \tilde{F} of $\alpha_{I_B} : I \rightarrow I_A \otimes I_B$ is $F^0_{I_B} : I \rightarrow F(I_A)(I_B)$, the component in I_B of the natural transformation $F^0 : I \rightarrow F(I_A) : \mathcal{B} \rightarrow \mathcal{C}$. These two arrows are equal since Axiom 2.11 for the natural monoidal transformation F^0 states exactly their equality.

For any objects a, a' in \mathcal{A} and b, b' in \mathcal{B} , the image by \tilde{F} of Diagram 12.22 is

$$\begin{array}{ccc}
(F(a)(b) \otimes F(a)(b')) \otimes (F(a')(b) \otimes F(a')(b')) & \xrightarrow{\cong} & (F(a)(b) \otimes F(a')(b)) \otimes (F(a)(b') \otimes F(a')(b')) \\
\downarrow F(a)_{b,b'}^2 \otimes F(a')_{b,b'}^2 & & \downarrow (F^2_{a,a'})_b \otimes (F^2_{a,a'})_{b'} \\
(F(a)(b \otimes b')) \otimes (F(a')(b \otimes b')) & & F(a \otimes a')(b) \otimes F(a \otimes a')(b') \\
\downarrow (F^2_{a,a'})_{b \otimes b'} & \nwarrow F(a \otimes a')_{b,b'}^2 & \\
F(a \otimes a')(b \otimes b') & &
\end{array}$$

which is commutative according to Axiom 2.10 for the monoidal natural transformation

$$F^2_{a,a'} : F(a) \square F(a') \rightarrow F(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}.$$

That the relations \sim obtained by the rules of expansion 12.23 are sent by \tilde{F} to commuting diagrams in \mathcal{C} , can be shown by induction according to the functoriality of tensor in \mathcal{C} .

Note that the functor $\bar{F} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ just defined admits a *symmetric strict monoidal* structure.

The functor En is defined on arrows of $SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$ as follows. Given any monoidal transformation between symmetric functors $\sigma : F \rightarrow G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, it is sent by En to the monoidal natural transformation $\bar{\sigma} : \bar{F} \rightarrow \bar{G} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ defined by induction on the structure of the elements of $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$ according to the following rules.

- $\bar{\sigma}_I : \bar{F}(I) \rightarrow \bar{G}(I)$ is the identity at $\bar{F}(I) = I = \bar{G}(I)$.
- For any objects a of \mathcal{A} and b of \mathcal{B} , $\bar{\sigma}_{a \otimes b} : \bar{F}(a \otimes b) \rightarrow \bar{G}(a \otimes b)$ is the arrow

$$\bar{F}(a \otimes b) \equiv F(a)(b) \xrightarrow{(\sigma_a)_b} G(a)(b) \equiv \bar{G}(a \otimes b)$$

where the transformation $\sigma_a : F(a) \rightarrow G(a) : \mathcal{B} \rightarrow \mathcal{C}$ is the component in a of σ .

- For any X, Y in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, the arrow

$$\bar{\sigma}_{X \otimes Y} : \bar{F}(X \otimes Y) \rightarrow \bar{G}(X \otimes Y)$$

is

$$\bar{F}(X \otimes Y) \equiv \bar{F}(X) \otimes \bar{F}(Y) \xrightarrow{\bar{\sigma}_X \otimes \bar{\sigma}_Y} \bar{G}(X) \otimes \bar{G}(Y) \equiv \bar{G}(X \otimes Y).$$

To show that the above assignments $\bar{\sigma}$ define a natural transformation $\bar{F} \rightarrow \bar{G}$, it is enough to show that for any $h : X \rightarrow Y$ in $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$, the diagram in \mathcal{C}

14.1

$$\begin{array}{ccc}
\bar{F}(X) & \xrightarrow{\bar{F}(h)} & \bar{F}(Y) \\
\bar{\sigma}_X \downarrow & & \downarrow \bar{\sigma}_Y \\
\bar{G}(X) & \xrightarrow{\bar{G}(h)} & \bar{G}(Y)
\end{array}$$

commutes. We shall check this now.

For any X, Y, Z in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, Diagram 14.1 for $h = ass_{X, Y, Z}$ is

$$\begin{array}{ccc} \bar{F}(X) \otimes (\bar{F}(Y) \otimes \bar{F}(Z)) & \xrightarrow{ass_{\bar{F}(X), \bar{F}(Y), \bar{F}(Z)}} & (\bar{F}(X) \otimes \bar{F}(Y)) \otimes \bar{F}(Z) \\ \bar{\sigma}_X \otimes (\bar{\sigma}_Y \otimes \bar{\sigma}_Z) \downarrow & & \downarrow (\bar{\sigma}_X \otimes \bar{\sigma}_Y) \otimes \bar{\sigma}_Z \\ \bar{G}(X) \otimes (\bar{G}(Y) \otimes \bar{G}(Z)) & \xrightarrow{ass_{\bar{G}(X), \bar{G}(Y), \bar{G}(Z)}} & (\bar{G}(X) \otimes \bar{G}(Y)) \otimes \bar{G}(Z) \end{array}$$

which commutes due to the naturality of ass .

For any X in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, Diagram 14.1 for $h = r_X$ is

$$\begin{array}{ccc} \bar{F}(X) \otimes I & \xrightarrow{r_{\bar{F}(X)}} & \bar{F}(X) \\ \bar{\sigma}_X \otimes 1 \downarrow & & \downarrow \bar{\sigma}_X \\ \bar{G}(X) \otimes I & \xrightarrow{r_{\bar{G}(X)}} & \bar{G}(X) \end{array}$$

which commutes due to the naturality of r .

Similarly, for any X in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, Diagram 14.1 for $h = l_X$ is commutative due to the naturality of l .

For any X, Y in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, Diagram 14.1 for $h = s_{X, Y}$ is

$$\begin{array}{ccc} \bar{F}(X) \otimes \bar{F}(Y) & \xrightarrow{s_{\bar{F}(X), \bar{F}(Y)}} & \bar{F}(Y) \otimes \bar{F}(X) \\ \bar{\sigma}_X \otimes \bar{\sigma}_Y \downarrow & & \downarrow \bar{\sigma}_Y \otimes \bar{\sigma}_X \\ \bar{G}(X) \otimes \bar{G}(Y) & \xrightarrow{s_{\bar{G}(X), \bar{G}(Y)}} & \bar{G}(Y) \otimes \bar{G}(X) \end{array}$$

which commutes due to the naturality of s .

From the above it is immediate that Diagram 14.1 also commutes for arrows h of form $\overline{ass}_{X, Y, Z}$, \bar{r}_X and \bar{l}_X .

For any object b of \mathcal{B} , Diagram 14.1 for $h = \alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ is

$$\begin{array}{ccc} I & \xrightarrow{F_b^0} & F(I_{\mathcal{A}})(b) \\ \parallel & & \downarrow (\sigma_{I_{\mathcal{A}}})_b \\ I & \xrightarrow{G_b^0} & G(I_{\mathcal{A}})(b) \end{array}$$

which is the evaluation in b of the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc} I & \xrightarrow{F^0} & F(I_{\mathcal{A}}) \\ & \searrow G^0 & \downarrow \sigma_{I_{\mathcal{A}}} \\ & & G(I_{\mathcal{A}}) \end{array}$$

which commutes according to Axiom 2.11 for the monoidal natural transformation σ .

For any object a of \mathcal{A} , Diagram 14.1 for $h = \beta_a : I \rightarrow a \otimes I_{\mathcal{B}}$ is

$$\begin{array}{ccc} I & \xrightarrow{F(a)^0} & F(a)(I_{\mathcal{B}}) \\ \parallel & & \downarrow (\sigma_a)_{I_{\mathcal{B}}} \\ I & \xrightarrow{G(a)^0} & G(a)(I_{\mathcal{B}}) \end{array}$$

which commutes according to Axiom 2.11 for the monoidal natural transformation $\sigma_a : F(a) \rightarrow G(a) : \mathcal{B} \rightarrow \mathcal{C}$.

For any objects a, a' of \mathcal{A} and b of \mathcal{B} , Diagram 14.1 for $h = \gamma_{a,a',b} : (a \otimes b) \otimes (a' \otimes b) \rightarrow (a \otimes a') \otimes b$ is

$$\begin{array}{ccc} F(a)(b) \otimes F(a')(b) & \xrightarrow{(F_{a,a'}^2)_b} & F(a \otimes a')(b) \\ (\sigma_a)_b \otimes (\sigma_{a'})_b \downarrow & & \downarrow (\sigma_{a \otimes a'})_b \\ G(a)(b) \otimes G(a')(b) & \xrightarrow{(G_{a,a'}^2)_b} & G(a \otimes a')(b) \end{array}$$

which is the evaluation in b of the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc} F(a) \square F(a') & \xrightarrow{F_{a,a'}^2} & F(a \otimes a') \\ \sigma_a \square \sigma_{a'} \downarrow & & \downarrow \sigma_{a \otimes a'} \\ G(a) \square G(a') & \xrightarrow{G_{a,a'}^2} & G(a \otimes a') \end{array}$$

which commutes according to Axiom 2.10 for the monoidal natural transformation σ .

For any objects a of \mathcal{A} and b, b' of \mathcal{B} , Diagram 14.1 for $h = \delta_{a,b,b'} : (a \otimes b) \otimes (a \otimes b') \rightarrow a \otimes (b \otimes b')$ is

$$\begin{array}{ccc} F(a)(b) \otimes F(a)(b') & \xrightarrow{F(a)_{b,b'}^2} & F(a)(b \otimes b') \\ (\sigma_a)_b \otimes (\sigma_a)_{b'} \downarrow & & \downarrow (\sigma_a)_{b \otimes b'} \\ G(a)(b) \otimes G(a)(b') & \xrightarrow{G(a)_{b,b'}^2} & G(a)(b \otimes b') \end{array}$$

which commutes according to Axiom 2.10 for the natural transformation $\sigma_a : F(a) \rightarrow G(a) : \mathcal{B} \rightarrow \mathcal{C}$.

For any arrow $f : a \rightarrow a'$ in \mathcal{A} and any object b in \mathcal{B} , Diagram 14.1 for $h = f \otimes b : a \otimes b \rightarrow a' \otimes b$ is

$$\begin{array}{ccc} F(a)(b) & \xrightarrow{F(f)_b} & F(a')(b) \\ (\sigma_a)_b \downarrow & & \downarrow (\sigma_{a'})_b \\ G(a)(b) & \xrightarrow{G(f)_b} & G(a')(b) \end{array}$$

which is the evaluation in b of the diagram in $[\mathcal{B}, \mathcal{C}]$

$$\begin{array}{ccc} F(a) & \xrightarrow{F(f)} & F(a') \\ \sigma_a \downarrow & & \downarrow \sigma_{a'} \\ G(a) & \xrightarrow{G(f)} & G(a') \end{array}$$

which commutes by naturality of σ .

For any object a in \mathcal{A} and any arrow $g : b \rightarrow b'$ in \mathcal{B} , Diagram 14.1 for $h = a \otimes g : a \otimes b \rightarrow a \otimes b'$ is

$$\begin{array}{ccc} F(a)(b) & \xrightarrow{F(a)(g)} & F(a)(b') \\ (\sigma_a)_b \downarrow & & \downarrow (\sigma_a)_{b'} \\ G(a)(b) & \xrightarrow{G(a)(g)} & G(a)(b') \end{array}$$

which commutes by naturality of $\sigma_a : F(a) \rightarrow G(a) : \mathcal{B} \rightarrow \mathcal{C}$.

So far we have shown that Diagram 14.1 commutes for any arrow h in $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$. That it is also the case for all arrows h of \mathcal{H} is now proved by induction.

For any object X and any arrow $f : Y \rightarrow Z$ in $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$, Diagram 14.1 for $h = X \otimes f : X \otimes Y \rightarrow X \otimes Z$ is

$$\begin{array}{ccc} \bar{F}(X) \otimes \bar{F}(Y) & \xrightarrow{1 \otimes \bar{F}(f)} & \bar{F}(X) \otimes \bar{F}(Z) \\ \bar{\sigma}_X \otimes \bar{\sigma}_Y \downarrow & & \downarrow \bar{\sigma}_X \otimes \bar{\sigma}_Z \\ \bar{G}(X) \otimes \bar{G}(Y) & \xrightarrow{1 \otimes \bar{G}(f)} & \bar{G}(X) \otimes \bar{G}(Z). \end{array}$$

According to the functoriality of tensor in \mathcal{C} , this diagram commutes if the diagram

$$\begin{array}{ccc} \bar{F}(Y) & \xrightarrow{\bar{F}(f)} & \bar{F}(Z) \\ \bar{\sigma}_Y \downarrow & & \downarrow \bar{\sigma}_Z \\ \bar{G}(Y) & \xrightarrow{\bar{G}(f)} & \bar{G}(Z) \end{array}$$

commutes. Similarly one shows that Diagram 14.1 for $h = f \otimes X : Y \otimes X \rightarrow Z \otimes X$ is commutative if Diagram 14.1 for $h = f$ commutes.

According to its inductive definition $\bar{\sigma} : \bar{F} \rightarrow \bar{G}$ is trivially monoidal between strict monoidal functors.

Let us give a universal characterisation of the \bar{F} and $\bar{\sigma}$ that we have just been defined.

Proposition 14.2 *Given any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, the symmetric monoidal*

functor \bar{F} is the unique strict one $\mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ that renders commutative the diagram in SMC

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta} & [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}] \\ & \searrow F & \downarrow [\mathcal{B}, \bar{F}] \\ & & [\mathcal{B}, \mathcal{C}]. \end{array}$$

Given any 2-cell $\sigma : F \rightarrow G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ in SMC , $\bar{\sigma} : \bar{F} \rightarrow \bar{G}$ is the unique 2-cell in SMC such that $[\mathcal{B}, \bar{\sigma}] * \eta = \sigma$.

PROOF: To see this, note that the commutation of the diagram in **Cat**

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta} & SMC(\mathcal{B}, \mathcal{A} \otimes \mathcal{B}) \\ & \searrow F & \downarrow SMC(\mathcal{B}, \bar{F}) \\ & & SMC(\mathcal{B}, \mathcal{C}) \end{array}$$

is equivalent to the conjunction of the following facts:

- (1): For any object a in \mathcal{A} , the underlying functors $F(a)$ and $\bar{F} \circ (\eta(a))$ are equal;
- (2): For any object a in \mathcal{A} , $(\bar{F} \circ (\eta(a)))^0 = F(a)^0$;
- (3): For any object a in \mathcal{A} , $(\bar{F} \circ (\eta(a)))^2 = F(a)^2$;
- (4): For any arrow $f : a \rightarrow a'$ in \mathcal{A} , the natural transformations $F(f)$ and $\bar{F} * (\eta(f))$ are equal.

Condition (1) is equivalent to the conjunction of the following two conditions:

- (1-1): For any objects a in \mathcal{A} and b in \mathcal{B} , $\bar{F}(a \otimes b) = F(a)(b)$;

and

- (1-2): For any object a in \mathcal{A} and any arrow $g : b \rightarrow b'$ in \mathcal{B} , $\bar{F}(a \otimes g) : \bar{F}(a \otimes b) \rightarrow \bar{F}(a \otimes b')$ is $F(a)(g) : F(a)(b) \rightarrow F(a)(b')$.

So if (1) holds, condition (4) is just that:

- (4') For any arrow $f : a \rightarrow a'$ in \mathcal{A} and any object b of \mathcal{B} , $F(f)_b : F(a)(b) \rightarrow F(a')(b)$ is equal to $\bar{F}(f \otimes b) : \bar{F}(a \otimes b) \rightarrow \bar{F}(a' \otimes b)$.

In the case when (1) holds and \bar{F} is strict, condition (2) is just equivalent to:

- (2'): For any object a in \mathcal{A} , \bar{F} sends $\beta_a : I \rightarrow a \otimes I_{\mathcal{B}}$ to $F(a)^0 : I_{\mathcal{C}} \rightarrow F(a)(I_{\mathcal{B}})$;

and condition (3) is equivalent to:

- (3'): For any objects a in \mathcal{A} and b, b' in \mathcal{B} , \bar{F} sends the arrow $\delta_{a,b,b'} : (a \otimes b) \otimes (a \otimes b') \rightarrow a \otimes (b \otimes b')$ to $(F_a)^2_{b,b'} : F_a(b) \otimes F_a(b') \rightarrow F_a(b \otimes b')$.

The commutation of the diagram in SMC of the proposition, for a strict \bar{F} , is therefore equivalent to the conjunction of the above conditions (1-1), (1-2), (2'), (3') and (4') and the two conditions:

- (5): $([\mathcal{B}, \bar{F}] \circ \eta)^0 = F^0$;

and

- (6): $([\mathcal{B}, \bar{F}] \circ \eta)^2 = F^2$.

Condition (5) is that:

- (5'): For any objects b in \mathcal{B} , \bar{F} sends the arrow $\alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ to $F^0_b : I \rightarrow F(I_{\mathcal{A}})(b)$.

Condition (6) is that:

- (6') For any objects b in \mathcal{B} and a, a' in \mathcal{A} , \bar{F} sends $\gamma_{a,a',b} : (a \otimes b) \otimes (a' \otimes b) \rightarrow (a \otimes a') \otimes b$ to $(F^2_{a,a'})_b : F_a(b) \otimes F_{a'}(b) \rightarrow F(a \otimes a')(b)$.

Now remark that according to its inductive definition, the monoidal functor \bar{F} , image of F by En , is the only strict one satisfying conditions (1-1), (1-2), (2'), (3'), (4'), (5') and (6') above.

Given monoidal transformations $\sigma : F \rightarrow G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, and $\bar{\sigma} : \bar{F} \rightarrow \bar{G}$, with \bar{F}, \bar{G} the respective images of F and G by En , that $[\mathcal{B}, \bar{\sigma}] * \eta = \sigma$ just means that for any object a in \mathcal{A} , the natural transformation $\sigma_a : F(a) \rightarrow G(a) : \mathcal{B} \rightarrow \mathcal{C}$ is $\bar{\sigma} * \eta(a)$, which is equivalent to the assertion that:

- (7) For any objects a in \mathcal{A} and b in \mathcal{B} , the arrow $\bar{\sigma}_{a \otimes b} : \bar{F}(a \otimes b) \rightarrow \bar{G}(a \otimes b)$ is $(\sigma_a)_b : Fa(b) \rightarrow Ga(b)$.

According to its inductive definition, the monoidal natural $\bar{\sigma} : \bar{F} \rightarrow \bar{G}$, image of any $\sigma : F \rightarrow G$ by En , is the unique one satisfying the condition (7) above.

■

One has this alternative characterisation from Proposition 14.2 and Lemma 10.1.

Remark 14.3 *Given any symmetric monoidal $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, the symmetric monoidal functor $\bar{F} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ is the only strict one that renders commutative the diagram in SMC*

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\eta^*} & [\mathcal{A}, \mathcal{A} \otimes \mathcal{B}] \\ & \searrow F^* & \downarrow [\mathcal{A}, \bar{F}] \\ & & [\mathcal{A}, \mathcal{C}]. \end{array}$$

Eventually one has also this last characterisation of the \bar{F} and $\bar{\sigma}$ using two diagrams in **Cat** rather than one in SMC .

Proposition 14.4 *Given any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, the symmetric monoidal functor $\bar{F} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ is the unique strict one such that the following two diagrams in **Cat** commute*

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta} & SMC(\mathcal{B}, \mathcal{A} \otimes \mathcal{B}) \\ & \searrow F & \downarrow SMC(\mathcal{B}, \bar{F}) \\ & & SMC(\mathcal{B}, \mathcal{C}) \end{array}$$

and

$$\begin{array}{ccc} \mathcal{B} & \xrightarrow{\eta^*} & SMC(\mathcal{A}, \mathcal{A} \otimes \mathcal{B}) \\ & \searrow F^* & \downarrow SMC(\mathcal{A}, \bar{F}) \\ & & SMC(\mathcal{A}, \mathcal{C}). \end{array}$$

PROOF: From the results of section 5 detailing the monoidal structure of F^* , one checks that the commutation of the second diagram in **Cat** in the proposition implies the following:

- For any objects b in \mathcal{B} , \bar{F} sends the arrow $\alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ to $F^0_b : I \rightarrow F(I_{\mathcal{A}})(b)$;
- For any objects b in \mathcal{B} and a, a' in \mathcal{A} , \bar{F} sends $\gamma_{a, a', b} : (a \otimes b) \otimes (a' \otimes b) \rightarrow (a \otimes a') \otimes b$ to $(F^2_{a, a'})_b : Fa(b) \otimes Fa'(b) \rightarrow F(a \otimes a')(b)$. ■

From Proposition 14.2 the following becomes immediate.

Corollary 14.5 *The assignments $(-)$ define a functor $En : SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$ which factorises as*

$$SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \xrightarrow{\cong} StrSMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \longrightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}),$$

where the functor on the left is an isomorphism.

It is also immediate from Proposition 14.2 that the composite mere functor $Rn \circ En$ is the identity of $SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$.

Let $(-)^{\nabla}$ denote the composite functor $En \circ Rn$. We show now the existence of a natural transformation $\epsilon : (-)^{\nabla} \rightarrow 1 : SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$.

For any symmetric monoidal $F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$, the monoidal natural transformation $\epsilon_F : F^{\nabla} \rightarrow F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ is defined by induction on the structure of the objects of $\mathcal{A} \otimes \mathcal{B}$ according to the following rules. We shall drop the subscript F and write simply ϵ when there is no ambiguity.

- For any objects a of \mathcal{A} and b of \mathcal{B} , $\epsilon_{a \otimes b}$ is the identity: $F^{\nabla}(a \otimes b) \equiv F(a \otimes b)$.
- ϵ_I is the arrow

$$F^{\nabla}(I) \equiv I \xrightarrow{F^0} F(I).$$

- For any objects X, Y of $\mathcal{A} \otimes \mathcal{B}$, $\epsilon_{X \otimes Y}$ is the arrow

$$F^{\nabla}(X) \otimes F^{\nabla}(Y) \xrightarrow{\epsilon_X \otimes \epsilon_Y} F(X) \otimes F(Y) \xrightarrow{F_{X,Y}^2} F(X \otimes Y).$$

To check the naturality of $\epsilon_F : F^{\nabla} \rightarrow F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$, one needs to show that for any arrow $h : X \rightarrow Y$ of $\mathcal{H}_{\mathcal{A}, \mathcal{B}}$ the following diagram in \mathcal{C} commutes

14.6

$$\begin{array}{ccc} F^{\nabla}(X) & \xrightarrow{F^{\nabla}(h)} & F^{\nabla}(Y) \\ \epsilon_X \downarrow & & \downarrow \epsilon_Y \\ F(X) & \xrightarrow{F(h)} & F(Y) \end{array}$$

where h also denotes the corresponding arrow of $\mathcal{A} \otimes \mathcal{B}$.

For any X, Y, Z in $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$, Diagram 14.6 for $h = ass_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ is

$$\begin{array}{ccc} F^{\nabla}(X \otimes (Y \otimes Z)) & \xrightarrow{F^{\nabla}(ass_{X,Y,Z})} & F^{\nabla}((X \otimes Y) \otimes Z) \\ \epsilon_{X \otimes (Y \otimes Z)} \downarrow & & \downarrow \epsilon_{(X \otimes Y) \otimes Z} \\ F(X \otimes (Y \otimes Z)) & \xrightarrow{F(ass_{X,Y,Z})} & F((X \otimes Y) \otimes Z) \end{array}$$

which is the external diagram in the pasting

$$\begin{array}{ccc} F^{\nabla}(X) \otimes (F^{\nabla}(Y) \otimes F^{\nabla}(Z)) & \xrightarrow{ass_{F^{\nabla}(X), F^{\nabla}(Y), F^{\nabla}(Z)}} & ((F^{\nabla}(X) \otimes F^{\nabla}(Y)) \otimes F^{\nabla}(Z)) \\ \epsilon_X \otimes (\epsilon_Y \otimes \epsilon_Z) \downarrow & & \downarrow (\epsilon_X \otimes \epsilon_Y) \otimes \epsilon_Z \\ F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{ass_{F(X), F(Y), F(Z)}} & ((F(X) \otimes F(Y)) \otimes F(Z)) \\ 1 \otimes F_{Y,Z}^2 \downarrow & & \downarrow F_{X,Y}^2 \otimes 1 \\ F(X) \otimes F(Y \otimes Z) & & (F(X \otimes Y)) \otimes F(Z) \\ F_{X,Y \otimes Z}^2 \downarrow & & \downarrow F_{X \otimes Y, Z}^2 \\ F(X \otimes (Y \otimes Z)) & \xrightarrow{F(ass_{X,Y,Z})} & F((X \otimes Y) \otimes Z) \end{array}$$

where the top square commutes by naturality of ass and the bottom one also does according to Axiom 2.6 for F .

For any X in $\mathcal{O}_{\mathcal{A},\mathcal{B}}$, Diagram 14.6 for $h = r_X : X \otimes I \rightarrow X$ is

$$\begin{array}{ccc} F^\nabla(X \otimes I) & \xrightarrow{F^\nabla(r_X)} & F^\nabla(X) \\ \epsilon_{X \otimes I} \downarrow & & \downarrow \epsilon_X \\ F(X \otimes I) & \xrightarrow{F(r_X)} & F(X) \end{array}$$

which is the external diagram in the pasting

$$\begin{array}{ccccc} F^\nabla(X) \otimes I & \xrightarrow{\tau_{F^\nabla(X)}} & F^\nabla(X) & & \\ \epsilon_X \otimes F^0 \downarrow & \searrow \epsilon_X \otimes 1 & & & \downarrow \epsilon_X \\ F(X) \otimes F(I) & \xleftarrow{1 \otimes F^0} & F(X) \otimes I & \xrightarrow{\tau_{F(X)}} & F(X) \\ F_{X,I}^2 \downarrow & & & & \downarrow \\ F(X \otimes I) & \xrightarrow{F(r_X)} & F(X) & & \end{array}$$

where the bottom left diagram commutes according to Axiom 2.7 for F whereas the right one commutes by naturality of r .

Similarly one shows that for any X in $\mathcal{O}_{\mathcal{A},\mathcal{B}}$, Diagram 14.6 for $h = l_X : X \otimes I \rightarrow X$ commutes according to Axiom 2.8 for F and the naturality of l .

For any X, Y in $\mathcal{O}_{\mathcal{A},\mathcal{B}}$, Diagram 14.6 for $h = s_{X,Y} : X \otimes Y \rightarrow Y \otimes X$ is

$$\begin{array}{ccc} F^\nabla(X \otimes Y) & \xrightarrow{F^\nabla(s_{X,Y})} & F^\nabla(Y \otimes X) \\ \epsilon_{X \otimes Y} \downarrow & & \downarrow \epsilon_{Y \otimes X} \\ F(X \otimes Y) & \xrightarrow{F(s_{X,Y})} & F(Y \otimes X) \end{array}$$

which is the external diagram in the pasting

$$\begin{array}{ccccc} F^\nabla(X) \otimes F^\nabla(Y) & \xrightarrow{s_{F^\nabla(X), F^\nabla(Y)}} & F^\nabla(Y) \otimes F^\nabla(X) & & \\ \epsilon_X \otimes \epsilon_Y \downarrow & & \downarrow \epsilon_Y \otimes \epsilon_X & & \\ F(X) \otimes F(Y) & \xrightarrow{s_{F(X), F(Y)}} & F(Y) \otimes F(X) & & \\ F_{X,Y}^2 \downarrow & & \downarrow F_{Y,X}^2 & & \\ F(X \otimes Y) & \xrightarrow{F(s_{X,Y})} & F(Y \otimes X) & & \end{array}$$

where the top square commutes by naturality of s and the bottom one also does according to Axiom 2.9 for F .

For any object b of \mathcal{B} , Diagram 14.6 for $h = \alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ is

$$\begin{array}{ccc} F^\nabla(I) & \xrightarrow{F^\nabla(\alpha_b)} & F^\nabla(I_{\mathcal{A}} \otimes b) \\ \epsilon_I \downarrow & & \downarrow \epsilon_{(I_{\mathcal{A}} \otimes b)} \\ F(I) & \xrightarrow{F(\alpha_b)} & F(I_{\mathcal{A}} \otimes b). \end{array}$$

We prove below that it commutes. The arrow $F(\alpha_b) \circ \epsilon_I$ is $I \xrightarrow{F^0} F(I) \xrightarrow{F(\alpha_b)} F(I_{\mathcal{A}} \otimes b)$. On the other hand the arrow $\epsilon_{(I_{\mathcal{A}} \otimes b)} \circ F^\nabla(\alpha_b)$ rewrites

$$I \equiv F^\nabla(I) \xrightarrow{F^\nabla(\alpha_b)} F^\nabla(I_{\mathcal{A}} \otimes b) \equiv F(I_{\mathcal{A}} \otimes b).$$

The functor F^∇ sends the arrow $\alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ of $\mathcal{A} \otimes \mathcal{B}$ to the arrow $Rn(F)_b^0 : I \rightarrow Rn(F)(I_{\mathcal{A}})(b)$ of \mathcal{C} , component in b of the monoidal natural transformation $Rn(F)^0 : I \rightarrow Rn(F)(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{C}$ which is part the monoidal structure of $Rn(F) : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$. $Rn(F)$ has been defined as the composite $\mathcal{A} \xrightarrow{\eta} [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}] \xrightarrow{[\mathcal{B}, F]} [\mathcal{B}, \mathcal{C}]$ thus $Rn(F)^0$ is the composite arrow in $[\mathcal{B}, \mathcal{C}]$

$$I \xrightarrow{[\mathcal{B}, F]^0} [\mathcal{B}, F](I_{[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}]}) \xrightarrow{[\mathcal{B}, F](\eta^0)} [\mathcal{B}, F](\eta(I_{\mathcal{A}})).$$

Now $[\mathcal{B}, F]^0 : I \rightarrow F \circ I_{[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}]}$ is pointwise in b the arrow $F^0 : I \rightarrow F(I_{\mathcal{A} \otimes \mathcal{B}})$ in \mathcal{C} , whereas $\eta^0 : I \rightarrow \eta(I_{\mathcal{A}}) : \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{B}$ takes value in b the arrow $\alpha_b : I \rightarrow I_{\mathcal{A}} \otimes b$ in $\mathcal{A} \otimes \mathcal{B}$ and thus $[\mathcal{B}, F](\eta^0)$ takes value in b the arrow $F(\alpha_b) : F(I) \rightarrow F(I_{\mathcal{A}} \otimes b)$ in \mathcal{C} .

For any object a of \mathcal{A} , Diagram 14.6 for $h = \beta_a : I \rightarrow a \otimes I_{\mathcal{B}}$ is

$$\begin{array}{ccc} F^\nabla(I) & \xrightarrow{F^\nabla(\beta_a)} & F^\nabla(a \otimes I_{\mathcal{B}}) \\ \epsilon_I \downarrow & & \downarrow \epsilon_{(a \otimes I_{\mathcal{B}})} \\ F(I) & \xrightarrow{F(\beta_a)} & F(a \otimes I_{\mathcal{B}}) \end{array}$$

and commutes as shown below. The arrow $F(\beta_a) \circ \epsilon_I$ rewrites

$$F^\nabla(I) \equiv I \xrightarrow{F^0} F(I) \xrightarrow{F(\beta_a)} F(a \otimes I_{\mathcal{B}})$$

whereas the arrow $\epsilon_{a \otimes I_{\mathcal{B}}} \circ F^\nabla(\beta_a)$ rewrites

$$F^\nabla(I) \xrightarrow{F^\nabla(\beta_a)} F^\nabla(a \otimes I_{\mathcal{B}}) \equiv F(a \otimes I_{\mathcal{B}}).$$

The functor F^∇ sends $\beta_a : I \rightarrow a \otimes I_{\mathcal{B}}$ to the arrow $(Rn(F)(a))^0 : I \rightarrow Rn(F)(a)(I_{\mathcal{B}})$ in \mathcal{C} , part of the monoidal structure of $Rn(F)(a)$. Now $Rn(F)(a)$ is the composite $\mathcal{B} \xrightarrow{\eta(a)} \mathcal{A} \otimes \mathcal{B} \xrightarrow{F} \mathcal{C}$ and thus the arrow $(Rn(F)(a))^0$ is the composite $I \xrightarrow{F^0} F(I) \xrightarrow{F(\beta_a)} F(a \otimes I_{\mathcal{B}})$ since $(\eta(a))^0 = \beta_a$.

For any objects a, a' of \mathcal{A} and b of \mathcal{B} , Diagram 14.6 for $h = \gamma_{a,a',b} : (a \otimes b) \otimes (a' \otimes b) \rightarrow (a \otimes a') \otimes b$ is

$$\begin{array}{ccc} F^\nabla((a \otimes b) \otimes (a' \otimes b)) & \xrightarrow{F^\nabla(\gamma_{a,a',b})} & F^\nabla((a \otimes a') \otimes b) \\ \downarrow \epsilon_{(a \otimes b) \otimes (a' \otimes b)} & & \downarrow \epsilon_{(a \otimes a') \otimes b} \\ F((a \otimes b) \otimes (a' \otimes b)) & \xrightarrow{F(\gamma_{a,a',b})} & F((a \otimes a') \otimes b) \end{array}$$

and commutes as shown below. The arrow $\epsilon_{(a \otimes b) \otimes (a' \otimes b)}$ is

$$F^\nabla((a \otimes b) \otimes (a' \otimes b)) \xlongequal{\quad} F(a \otimes b) \otimes F(a' \otimes b) \xrightarrow{F^2_{a \otimes b, a' \otimes b}} F((a \otimes b) \otimes (a' \otimes b)).$$

One has $F^\nabla((a \otimes a') \otimes b) = F((a \otimes a') \otimes b)$ and the arrow $\epsilon_{(a \otimes a') \otimes b}$ is the identity at $F((a \otimes a') \otimes b)$. The functor F^∇ sends $\gamma_{a,a',b}$ to

$$((Rn(F))_{a,a'}^2)_b : Rn(F)(a)(b) \otimes Rn(F)(a')(b) \rightarrow Rn(F)(a \otimes a')(b)$$

which is the component in b of the monoidal natural transformation

$$Rn(F)_{a,a'}^2 : Rn(F)(a) \square Rn(F)(a') \rightarrow Rn(F)(a \otimes a') : \mathcal{B} \rightarrow \mathcal{C}.$$

This last one is the composite

$$[\mathcal{B}, F](\eta(a)) \square [\mathcal{B}, F](\eta(a')) \xrightarrow{[\mathcal{B}, F]_{\eta(a), \eta(a')}^2} [\mathcal{B}, F](\eta(a) \square \eta(a')) \xrightarrow{[\mathcal{B}, F](\eta_{a,a'}^2)} [\mathcal{B}, F](\eta(a \otimes a'))$$

which component in b is the arrow

$$F(a \otimes b) \otimes F(a' \otimes b) \xrightarrow{F^2_{a \otimes b, a' \otimes b}} F((a \otimes b) \otimes (a' \otimes b)) \xrightarrow{F(\gamma_{a,a',b})} F((a \otimes a') \otimes b).$$

For any objects a of \mathcal{A} and b, b' of \mathcal{B} , Diagram 14.6 for $h = \delta_{a,b,b'} : (a \otimes b) \otimes (a \otimes b') \rightarrow a \otimes (b \otimes b')$ is

$$\begin{array}{ccc} F^\nabla((a \otimes b) \otimes (a \otimes b')) & \xrightarrow{F^\nabla(\delta_{a,b,b'})} & F^\nabla(a \otimes (b \otimes b')) \\ \downarrow \epsilon_{(a \otimes b) \otimes (a \otimes b')} & & \downarrow \epsilon_{a \otimes (b \otimes b')} \\ F((a \otimes b) \otimes (a \otimes b')) & \xrightarrow{F(\delta_{a,b,b'})} & F(a \otimes (b \otimes b')), \end{array}$$

which commutes as shown below. $F^\nabla(a \otimes (b \otimes b'))$ is $F(a \otimes (b \otimes b'))$ and $\epsilon_{a \otimes (b \otimes b')}$ is the identity at $F(a \otimes (b \otimes b'))$. The arrow $\epsilon_{(a \otimes b) \otimes (a \otimes b')}$ is

$$F^\nabla((a \otimes b) \otimes (a \otimes b')) \xlongequal{\quad} F(a \otimes b) \otimes F(a \otimes b') \xrightarrow{F^2_{a \otimes b, a \otimes b'}} F((a \otimes b) \otimes (a \otimes b')) .$$

The functor F^∇ sends the arrow $\delta_{a,b,b'}$ to the arrow

$$(Rn(F)(a))_{b,b'}^2 : Rn(F)(a)(b) \otimes Rn(F)(a)(b') \rightarrow Rn(F)(a)(b \otimes b')$$

in \mathcal{C} . Since the functor $Rn(F)(a) : \mathcal{B} \rightarrow \mathcal{C}$ is the composite $\mathcal{B} \xrightarrow{\eta(a)} \mathcal{A} \otimes \mathcal{B} \xrightarrow{F} \mathcal{C}$, the arrow

$(Rn(F)(a))_{b,b'}^2$ is

$$\begin{array}{c}
Rn(F)(a)(b) \otimes Rn(F)(a)(b') \\
\parallel \\
F(a \otimes b) \otimes F(a \otimes b') \\
\downarrow F_{a \otimes b, a \otimes b'}^2 \\
F((a \otimes b) \otimes (a \otimes b')) \\
\downarrow F(\delta_{a,b,b'}) \\
F(a \otimes (b \otimes b')) \\
\parallel \\
Rn(F)(a)(b \otimes b').
\end{array}$$

For any arrow $f : a \rightarrow a'$ in \mathcal{A} and any object b of \mathcal{B} , Diagram 14.6 for $h = f \otimes b : a \otimes b \rightarrow a' \otimes b$ is

$$\begin{array}{ccc}
F^\nabla(a \otimes b) & \xrightarrow{F^\nabla(f \otimes 1)} & F^\nabla(a' \otimes b) \\
\epsilon_{a \otimes b} \downarrow & & \downarrow \epsilon_{a' \otimes b} \\
F(a \otimes b) & \xrightarrow{F(f \otimes 1)} & F(a' \otimes b)
\end{array}$$

and commutes since $\epsilon_{a \otimes b}$ (resp. $\epsilon_{a' \otimes b}$) is the identity at $F(a \otimes b)$ (resp. at $F(a' \otimes b)$) and straightforward computation shows that the $F^\nabla(f \otimes b)$ is

$$F^\nabla(a \otimes b) \xlongequal{\quad} F(a \otimes b) \xrightarrow{F(f \otimes 1)} F(a' \otimes b) \xlongequal{\quad} F^\nabla(a \otimes b).$$

For any object a of \mathcal{A} and any arrow $g : b \rightarrow b'$ in \mathcal{B} , Diagram 14.6 for $h = 1 \otimes g : a \otimes b \rightarrow a \otimes b'$ is

$$\begin{array}{ccc}
F^\nabla(a \otimes b) & \xrightarrow{F^\nabla(1 \otimes g)} & F^\nabla(a \otimes b') \\
\epsilon_{a \otimes b} \downarrow & & \downarrow \epsilon_{a \otimes b'} \\
F(a \otimes b) & \xrightarrow{F(1 \otimes g)} & F(a \otimes b')
\end{array}$$

which commutes since the arrows ϵ above are identities and the image by F^∇ of $1 \otimes g : a \otimes b \rightarrow a \otimes b'$ is $Rn(F)(a)(g)$ which is $F(1 \otimes g)$.

So far we have shown that for any arrow h in $\mathcal{H}_1 \cup \mathcal{H}_2 \cup \mathcal{H}_3$, Diagram 14.6 commutes. Now we prove by induction that this is the case for all arrows h in \mathcal{H} .

For any object X in $\mathcal{O}_{\mathcal{A},\mathcal{B}}$ and any arrow $f : Y \rightarrow Z$ in $\mathcal{H}_{\mathcal{A},\mathcal{B}}$, Diagram 14.6 for $h = X \otimes f : X \otimes Y \rightarrow X \otimes Z$ is

$$\begin{array}{ccc}
F^\nabla(X \otimes Y) & \xrightarrow{F^\nabla(X \otimes f)} & F^\nabla(X \otimes Z) \\
\epsilon_{X \otimes Y} \downarrow & & \downarrow \epsilon_{X \otimes Z} \\
F(X \otimes Y) & \xrightarrow{F(X \otimes f)} & F(X \otimes Z)
\end{array}$$

which is the external diagram in the pasting

$$\begin{array}{ccc}
F^\nabla(X) \otimes F^\nabla(Y) & \xrightarrow{1 \otimes F^\nabla(f)} & F^\nabla(X) \otimes F^\nabla(Z) \\
\epsilon_X \otimes \epsilon_Y \downarrow & & \downarrow \epsilon_X \otimes \epsilon_Z \\
F(X) \otimes F(Y) & \xrightarrow{1 \otimes F(f)} & F(X) \otimes F(Z) \\
F_{X,Y}^2 \downarrow & & \downarrow F_{X,Z}^2 \\
F(X \otimes Y) & \xrightarrow{F(X \otimes f)} & F(X \otimes Z)
\end{array}$$

By functoriality of the tensor in \mathcal{C} , the top diagram commutes if Diagram 14.6 commutes for $h = f$. The bottom one commutes by naturality of F^2 .

Similarly one shows that Diagram 14.6 for $h = f \otimes X : Y \otimes X \rightarrow Z \otimes X$ commutes providing the commutation of Diagram 14.6 for $h = f$.

We show now that the natural transformation $\epsilon : F^\nabla \rightarrow F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ is monoidal.

It satisfies Axiom 2.10 i.e. for any X, Y in $\mathcal{A} \otimes \mathcal{B}$, the diagram in \mathcal{C}

$$\begin{array}{ccc}
F^\nabla(X) \otimes F^\nabla(Y) & \xrightarrow{(F^\nabla)_{X,Y}^2} & F^\nabla(X \otimes Y) \\
\epsilon_X \otimes \epsilon_Y \downarrow & & \downarrow \epsilon_{(X \otimes Y)} \\
F(X) \otimes F(Y) & \xrightarrow{F_{X,Y}^2} & F(X \otimes Y)
\end{array}$$

commutes. This is the case since F^∇ is strict and according to the inductive definition of ϵ .

The natural transformation ϵ satisfies Axiom 2.11, i.e. the diagram in \mathcal{C}

$$\begin{array}{ccc}
I & \xrightarrow{(F^\nabla)^0} & F^\nabla(I) \\
& \searrow F^0 & \downarrow \epsilon_I \\
& & F(I)
\end{array}$$

commutes. This holds since F^∇ is strict and by the definition of ϵ at I .

Eventually, we show that the collection of monoidal transformations $\epsilon_F : F^\nabla \rightarrow F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ for the symmetric monoidal functors $F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$, constitutes a natural transformation $En \circ Rn \rightarrow 1 : SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$. Given any monoidal natural transformation $\sigma : F \rightarrow G : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ between symmetric monoidal functors, we check by induction on the objects of $\mathcal{O}_{\mathcal{A}, \mathcal{B}}$ that the following diagram in $SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$ commutes

14.7

$$\begin{array}{ccc}
F^\nabla & \xrightarrow{\sigma^\nabla} & G^\nabla \\
\epsilon_F \downarrow & & \downarrow \epsilon_G \\
F & \xrightarrow{\sigma} & G
\end{array}$$

Pointwise in I , Diagram 14.7 is

$$\begin{array}{ccc} I & \xlongequal{\quad} & I \\ F^0 \downarrow & & \downarrow G^0 \\ F(I) & \xrightarrow{\sigma_I} & G(I) \end{array}$$

which commutes according to Axiom 2.11 for the monoidal σ .

For any objects a in \mathcal{A} and b in \mathcal{B} , Diagram 14.7 pointwise in $a \otimes b$ commutes since $F^\nabla(a \otimes b) = F(a \otimes b)$, $G^\nabla(a \otimes b) = G(a \otimes b)$, the components of ϵ_F and ϵ_G in $a \otimes b$ are identities and $\sigma^\nabla_{a \otimes b}$ is

$$F^\nabla(a \otimes b) \xlongequal{\quad} F(a \otimes b) \xrightarrow{\sigma_{a \otimes b}} G(a \otimes b) \xlongequal{\quad} G^\nabla(a \otimes b) .$$

For any X and Y , Diagram 14.7, pointwise in $X \otimes Y$, is the external diagram in the pasting

$$\begin{array}{ccc} F^\nabla(X) \otimes F^\nabla(Y) & \xrightarrow{\sigma_X^\nabla \otimes \sigma_Y^\nabla} & G^\nabla(X) \otimes G^\nabla(Y) \\ (\epsilon_F)_X \otimes (\epsilon_F)_Y \downarrow & & \downarrow (\epsilon_G)_X \otimes (\epsilon_G)_Y \\ F(X) \otimes F(Y) & \xrightarrow{\sigma_X \otimes \sigma_Y} & G(X) \otimes G(Y) \\ F_{X,Y}^2 \downarrow & & \downarrow G_{X,Y}^2 \\ F(X \otimes Y) & \xrightarrow{\sigma_{X \otimes Y}} & G(X \otimes Y) . \end{array}$$

The top diagram commutes if Diagram 14.7 commutes pointwise in X and Y and the bottom one commutes according to Axiom 2.10 for σ .

Remark 14.8 For any strict (respectively strong) symmetric monoidal functor $F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$, ϵ_F is the identity (respectively an isomorphism).

Proposition 14.9 The functor $Rn : SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \rightarrow SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$ is right adjoint to $En : SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$.

PROOF: The unit of this adjunction is the identity and the counit is given in any $F : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{C}$ by $\epsilon_F : (En \circ Rn)(F) \rightarrow F$. The two triangular equalities amount then to the facts that

$$\epsilon * En : En \circ Rn \circ En \rightarrow En$$

and

$$Rn * \epsilon : Rn \circ En \circ Rn \rightarrow Rn$$

are identities. That $\epsilon * En$ is the identity results from Remark 14.8. That $Rn * \epsilon$ is the identity is immediate from the definition of ϵ . ■

As the composite of two strict symmetric monoidal functors, the functor $Rn : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ is strict.

Proposition 14.10 One has an adjunction $En \dashv Rn : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ in SMC where Rn is strict and the underlying adjunction in **Cat** is the one described in Proposition 14.9.

PROOF: According to Kelly's result (see 2.13), it is enough to show the two points below:

- (1) For any symmetric monoidal functors $F, G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$ the arrow

$$En(F \square G) \xrightarrow{En(u_F \square u_G)} En(Rn En(F) \square Rn En(G)) \xrightarrow{En(Rn^2_{En(F), En(G)})} En Rn(En(F) \square En(G)) \xrightarrow{\epsilon_{En(F) \square En(G)}} En(F) \square En(G)$$

where u denotes the unit of the adjunction $En \dashv Rn$, is an isomorphism;
- (2) The arrow

$$En(I) \xrightarrow{En(Rn^0)} EnRn(I) \xrightarrow{\epsilon_I} I$$

is an isomorphism.

(1) holds since the unit u of the adjunction is the identity, Rn is strict and $\epsilon_{En(F) \square En(G)}$ is an isomorphism by Remark 14.8 since $En(F)$ and $En(G)$ are strict and thus $En(F) \square En(G)$ is strong.

(2) holds since Rn is strict and since the unit of $[A \otimes B, C]$ is strong and thus by Remark 14.8 the arrow $\epsilon_I : EnRn(I) \rightarrow I$ is an isomorphism. \blacksquare

15 The tensor 2-functor $SMC \times SMC \rightarrow SMC$

In this section, the mapping sending any symmetric monoidal categories A and B to their tensor $A \otimes B$ is extended to a 2-functor $\underline{Ten} : SMC \times SMC \rightarrow SMC$.

From Proposition 14.2 and its corollary 14.5, one has for any A, B and C in SMC , an isomorphism in **Cat**

$$SMC(A, [B, C]) \cong StrSMC(A \otimes B, C).$$

Actually for any given A and B , this isomorphism is 2-natural in the argument C between 2-functors $StrSMC \rightarrow \mathbf{Cat}$. Therefore there is a unique way of extending the assignments $(A, B) \mapsto A \otimes B$ into a 2-functor $SMC \times SMC \rightarrow StrSMC$ which makes the collection of isomorphisms above also 2-natural in A in B , i.e. such that this collection defines a 2-natural transformation between 2-functors $SMC \times SMC \times StrSMC \rightarrow \mathbf{Cat}$. We shall call this extension the *tensor* 2-functor on SMC . We write \underline{Ten} for the corresponding functor $SMC \times SMC \rightarrow SMC$ and also extend the use the binary operation symbol \otimes to 1-cells and 2-cells to denote its images.

By a simple application of the Yoneda Lemma, one obtains the concrete description of this tensor, as follows.

15.1 For any $F : A \rightarrow C$ and any B in SMC , the 1-cell $\underline{Ten}(F, 1_B) : A \otimes B \rightarrow C \otimes B$, which we also write $F \otimes B$, is the image by En of $A \xrightarrow{F} C \xrightarrow{\eta} [B, C \otimes B]$.

15.2 For any $G : B \rightarrow D$ and any A in SMC , the 1-cell $\underline{Ten}(1_A, G) : A \otimes B \rightarrow A \otimes D$, which we also write $A \otimes G$, is the image by En of $A \xrightarrow{\eta} [D, A \otimes D] \xrightarrow{[G, 1]} [B, A \otimes D]$.

15.3 For any B and any 2-cell $\sigma : F \rightarrow F' : A \rightarrow C$ in SMC , the 2-cell

$$\underline{Ten}(\sigma, B) : F \otimes B \rightarrow F' \otimes B : A \otimes B \rightarrow C \otimes B,$$

which we also write $\sigma \otimes B$, is the image by En of the 2-cell

$$\begin{array}{ccc}
& F & \\
& \downarrow \sigma & \\
A & \xrightarrow{\quad} & C \xrightarrow{\eta} [B, C \otimes B] \\
& \uparrow F' &
\end{array}$$

15.4 For any \mathcal{A} and any 2-cell $\tau : G \rightarrow G' : \mathcal{B} \rightarrow \mathcal{D}$ in SMC , the 2-cell

$$\underline{Ten}(\mathcal{A}, \tau) : \mathcal{A} \otimes G \rightarrow \mathcal{A} \otimes G' : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{A} \otimes \mathcal{D},$$

which we also write $\mathcal{A} \otimes \tau$, is the image by En of the 2-cell

$$\begin{array}{ccc} & [G, 1] & \\ \swarrow & \parallel & \searrow \\ \mathcal{A} \xrightarrow{\eta} [\mathcal{D}, \mathcal{A} \otimes \mathcal{D}] & [\tau, 1] & [\mathcal{B}, \mathcal{A} \otimes \mathcal{D}] \\ \searrow & \parallel & \swarrow \\ & [G', 1] & \end{array}$$

Note that for any \mathcal{A}, \mathcal{B} and \mathcal{C} in SMC , the 2-functor $\underline{Ten}(-, \mathcal{B}) : SMC \rightarrow SMC$ has component in \mathcal{A} and \mathcal{C} , a functor $\underline{Ten}(-, \mathcal{B})_{\mathcal{A}, \mathcal{C}} : SMC(\mathcal{A}, \mathcal{C}) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C} \otimes \mathcal{B})$ which admits the symmetric monoidal structure

$$[\mathcal{A}, \mathcal{C}] \xrightarrow{[1, \eta]} [\mathcal{A}, [\mathcal{B}, \mathcal{C} \otimes \mathcal{B}]] \xrightarrow{En} [\mathcal{A} \otimes \mathcal{B}, \mathcal{C} \otimes \mathcal{B}].$$

Similarly for any $\mathcal{A}, \mathcal{B}, \mathcal{D}$ in SMC , the 2-functor $\underline{Ten}(\mathcal{A}, -) : SMC \rightarrow SMC$ has for component in \mathcal{B} and \mathcal{D} , a functor $\underline{Ten}(\mathcal{A}, -)_{\mathcal{B}, \mathcal{D}} : SMC(\mathcal{B}, \mathcal{D}) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{A} \otimes \mathcal{D})$ which admits the symmetric monoidal structure

$$[\mathcal{B}, \mathcal{D}] \xrightarrow{[-, \mathcal{A} \otimes \mathcal{D}]} [[\mathcal{D}, \mathcal{A} \otimes \mathcal{D}], [\mathcal{B}, \mathcal{A} \otimes \mathcal{D}]] \xrightarrow{[\eta, 1]} [\mathcal{A}, [\mathcal{B}, \mathcal{A} \otimes \mathcal{D}]] \xrightarrow{En} [\mathcal{A} \otimes \mathcal{B}, \mathcal{A} \otimes \mathcal{D}].$$

16 Naturality issues for Rn and En

This section tackles the questions of the naturalities of the collections of arrows $Rn_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ and $En_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$ in SMC .

From the definition of the tensor in SMC the underlying functors $SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$ of the 1-cells $En_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ of SMC define a 2-natural transformation between 2-functors $SMC^{op} \times SMC^{op} \times StrSMC \rightarrow \mathbf{Cat}$. Be cautious here that the domain for the third argument \mathcal{C} of the considered 2-functors is $StrSMC$, the 2-category with 1-cells *strict* functors. This statement will be further improved by Lemmas 16.7, 16.8, 16.9 below as the collections of 1-cells $En_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$ defines a 2-natural transformation between SMC -valued 2-functors with domains $SMC^{op} \times SMC^{op} \times StrSMC$.

Lemma 16.1 For any \mathcal{A} and \mathcal{B} , the collection of 1-cells in SMC

$$Rn_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

is 2-natural in \mathcal{C} .

PROOF: The 1-cell $Rn_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ is the composite

$$[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \xrightarrow{[\mathcal{B}, -]} [[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}, \mathcal{C}]] \xrightarrow{[\eta, 1]} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

in SMC where the collection of arrows $[\mathcal{B}, -]_{\mathcal{A} \otimes \mathcal{B}, \mathcal{C}}$ is 2-natural in \mathcal{C} according to Lemma 9.10 and the collection of $[\eta, [\mathcal{B}, \mathcal{C}]] : [[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ is 2-natural in \mathcal{C} according to the 2-functoriality of \underline{Hom} . ■

Lemma 16.2 *The diagram in SMC*

$$\begin{array}{ccc}
& [\mathcal{C}, \mathcal{C}'] & \\
\swarrow [\mathcal{A} \otimes \mathcal{B}, -] & & \searrow [\mathcal{B}, -] \\
[[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}], [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}']] & & [[\mathcal{B}, \mathcal{C}], [\mathcal{B}, \mathcal{C}']] \\
\downarrow [1, Rn] & & \downarrow [\mathcal{A}, -] \\
[[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}], [\mathcal{A}, [\mathcal{B}, \mathcal{C}']]] & \xleftarrow{[Rn, 1]} & [[\mathcal{A}, [\mathcal{B}, \mathcal{C}]], [\mathcal{A}, [\mathcal{B}, \mathcal{C}']]]
\end{array}$$

commutes for any \mathcal{A} , \mathcal{B} , \mathcal{C} and \mathcal{C}' .

PROOF: Since all the functors involved in the considered diagram are strict, it is enough to show that the underlying diagram in **Cat**

$$\begin{array}{ccc}
& SMC(\mathcal{C}, \mathcal{C}') & \\
\swarrow \underline{Hom}(\mathcal{A} \otimes \mathcal{B}, -) & & \searrow \underline{Hom}(\mathcal{B}, -) \\
SMC([\mathcal{A} \otimes \mathcal{B}, \mathcal{C}], [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}']) & & SMC([\mathcal{B}, \mathcal{C}], [\mathcal{B}, \mathcal{C}']) \\
\downarrow SMC(1, Rn) & & \downarrow \underline{Hom}(\mathcal{A}, -) \\
SMC([\mathcal{A} \otimes \mathcal{B}, \mathcal{C}], [\mathcal{A}, [\mathcal{B}, \mathcal{C}']]) & \xleftarrow{SMC(Rn, 1)} & SMC([\mathcal{A}, [\mathcal{B}, \mathcal{C}]], [\mathcal{A}, [\mathcal{B}, \mathcal{C}']]).
\end{array}$$

commutes, which amounts to the 2-naturality in \mathcal{C} of the collection $Rn_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$ in SMC , that was established in Lemma 16.1. \blacksquare

Lemma 16.3 *For any \mathcal{B} and \mathcal{C} , the collection of 1-cells in SMC*

$$Rn_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

is 2-natural in \mathcal{A} .

PROOF: Given any 1-cell $F : \mathcal{A}' \rightarrow \mathcal{A}$, consider the pasting in SMC

$$\begin{array}{ccccc}
[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{[\mathcal{B}, -]} & [[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}, \mathcal{C}]] & \xrightarrow{[\eta, 1]} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \\
\downarrow [F \otimes 1, 1] & & \downarrow [[\mathcal{B}, F \otimes 1], 1] & & \downarrow [F, 1] \\
[\mathcal{A}' \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{[\mathcal{B}, -]} & [[\mathcal{B}, \mathcal{A}' \otimes \mathcal{B}], [\mathcal{B}, \mathcal{C}]] & \xrightarrow{[\eta, 1]} & [\mathcal{A}', [\mathcal{B}, \mathcal{C}]]
\end{array}$$

in which the left diagram commutes according to Corollary 9.9 and the right one commutes according to the definition of $F \otimes 1$.

One easily adapts the previous argument to show that for any 2-cell $\sigma : F \rightarrow F' : \mathcal{A}' \rightarrow \mathcal{A}$ in SMC , one has the equality of 2-cells $[\sigma, 1] * Rn = Rn * [\sigma \otimes 1, 1]$ in SMC . \blacksquare

Lemma 16.4 *For any \mathcal{A} and \mathcal{C} , the collection of 1-cells in SMC*

$$Rn_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$$

is 2-natural in \mathcal{B} .

PROOF: Given any 1-cell $G : \mathcal{B}' \rightarrow \mathcal{B}$, consider the pasting in SMC

$$\begin{array}{ccccc}
[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{[\mathcal{B}, -]} & [[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}, \mathcal{C}]] & \xrightarrow{[\eta, 1]} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \\
\downarrow [1 \otimes G, 1] & \searrow [\mathcal{B}', -] & \downarrow [1, [G, 1]] & & \downarrow [1, [G, 1]] \\
& & [[\mathcal{B}', \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}', \mathcal{C}]] & \xrightarrow{[[G, 1], 1]} & [[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}', \mathcal{C}]] \\
& & \downarrow [[1, 1 \otimes G], 1] & \searrow [\eta, 1] & \\
[\mathcal{A} \otimes \mathcal{B}', \mathcal{C}] & \xrightarrow{[\mathcal{B}', -]} & [[\mathcal{B}', \mathcal{A} \otimes \mathcal{B}'], [\mathcal{B}', \mathcal{C}]] & \xrightarrow{[\eta, 1]} & [\mathcal{A}, [\mathcal{B}', \mathcal{C}]]
\end{array}$$

Here all the diagrams involved commute: the top left one commutes according to Lemma 9.11, the bottom left one commutes according to Corollary 9.9 and the bottom right one according to the definition of $1 \otimes G$.

One can adapt the previous argument to show for any 2-cell $\sigma : G \rightarrow G' : \mathcal{B}' \rightarrow \mathcal{B}$, in SMC , the equality of the 2-cells $Rn * [[\mathcal{A} \otimes \sigma], \mathcal{C}] = [\mathcal{A}, [\sigma, \mathcal{C}]] * Rn$ in SMC . ■

Consider any objects \mathcal{A} and \mathcal{B} and any 1-cell $F : \mathcal{C} \rightarrow \mathcal{C}'$ in SMC . Lemma 16.1 states that there is an identity 2-cell in SMC

$$\begin{array}{ccc}
[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \\
[\mathcal{A} \otimes \mathcal{B}, F] \downarrow & = & \downarrow [\mathcal{A}, [\mathcal{B}, F]] \\
[\mathcal{A} \otimes \mathcal{B}, \mathcal{C}'] & \xrightarrow{Rn} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}']].
\end{array}$$

Its “mate”, that we denote Ξ_F , is a 2-cell in SMC

16.5

$$\begin{array}{ccc}
[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \\
[\mathcal{A}, [\mathcal{B}, F]] \downarrow & \xRightarrow{\Xi_F} & \downarrow [\mathcal{A} \otimes \mathcal{B}, F] \\
[\mathcal{A}, [\mathcal{B}, \mathcal{C}']] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}'].
\end{array}$$

More explicitly Ξ_F is the following pasting in SMC

$$\begin{array}{ccccc}
& & 1 & & \\
& \swarrow & \parallel & \searrow & \\
[\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \\
& & \downarrow [\mathcal{A} \otimes \mathcal{B}, F] & = & \downarrow [\mathcal{A}, [\mathcal{B}, F]] \\
& & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}'] & \xrightarrow{Rn} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}']] \\
& & & & \downarrow \epsilon \\
& & & & 1
\end{array}$$

where the top identity 2-cell and ϵ are respectively the unit and counit of the adjunctions $En \dashv Rn$ in the 2-category SMC .

Considering the 2-cell above, observe that for any symmetric monoidal $G : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{C}]$, the component in G of Ξ_F is $\epsilon_{F \circ En(G)}$, which is the identity if F is strict. Let us take note of this.

Remark 16.6 The 2-cell Ξ_F is the identity for any strict F .

Also and still according to Lemma 16.1, for any 2-cell $\sigma : F \rightarrow G : \mathcal{C} \rightarrow \mathcal{C}'$, the two pastings in SMC

$$\begin{array}{ccc}
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \\
 \downarrow [\mathcal{A}, [\mathcal{B}, F]] & \nearrow \Xi_F & \downarrow [1, F] \\
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}']] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}']
 \end{array}
 \quad
 \begin{array}{ccc}
 & & [1, G] \\
 & \xleftarrow{[1, \sigma]} & \\
 & &
 \end{array}$$

and

$$\begin{array}{ccc}
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \\
 \downarrow [\mathcal{A}, [\mathcal{B}, F]] & \nearrow \Xi_G & \downarrow [\mathcal{A} \otimes \mathcal{B}, G] \\
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}']] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}']
 \end{array}
 \quad
 \begin{array}{ccc}
 & & [1, [1, \sigma]] \\
 & \xleftarrow{[1, [1, \sigma]]} & \\
 & &
 \end{array}$$

are equal.

As a consequence of this, one has the following result.

Lemma 16.7 For any \mathcal{A} and \mathcal{B} , the collection of 1-cells $En_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$ in SMC defines a 2-natural transformation in the argument \mathcal{C} between the restrictions of the 2-functors $\underline{Hom}(\mathcal{A}, -) \circ \underline{Hom}(\mathcal{B}, -)$ and $\underline{Hom}(\mathcal{A} \otimes \mathcal{B}, -)$ to the 2-category $StrSMC$.

Lemma 16.8 For any \mathcal{B} and \mathcal{C} , the collection of 1-cells $En_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$ of SMC defines a 2-natural transformation in the argument \mathcal{A}

$$\underline{Hom}(-, [\mathcal{B}, \mathcal{C}]) \rightarrow \underline{Hom}(-, \mathcal{C}) \circ \underline{Ten}(-, \mathcal{B}) : SMC^{op} \rightarrow SMC.$$

PROOF: Consider any 1-cell $F : \mathcal{A} \rightarrow \mathcal{A}'$ in SMC . Let us see that the diagram

$$\begin{array}{ccc}
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \\
 \downarrow [F, 1] & & \downarrow [F \otimes 1, 1] \\
 [\mathcal{A}', [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A}' \otimes \mathcal{B}, \mathcal{C}]
 \end{array}$$

in SMC commutes. According to Lemma 16.3, the following diagram in SMC commutes

$$\begin{array}{ccc}
 [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \\
 \downarrow [F \otimes 1, 1] & & \downarrow [F, 1] \\
 [\mathcal{A}' \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}', [\mathcal{B}, \mathcal{C}]]
 \end{array}$$

and the corresponding identity 2-cell has for mate

$$En \circ [F, 1] \rightarrow [F \otimes 1, 1] \circ En : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A}' \otimes \mathcal{B}, \mathcal{C}]$$

the 2-cell $\epsilon * ([F \otimes 1, 1] \circ En)$. In any object H of $[\mathcal{A}, [\mathcal{B}, \mathcal{C}]]$, this monoidal natural transformation has component $\epsilon_{En(H) \circ (F \otimes 1)}$ which is an identity.

Consider now any 2-cell $\sigma : F \rightarrow G : \mathcal{A}' \rightarrow \mathcal{A}$ in SMC . According to Lemma 16.3, the 2-cells

$$\begin{array}{c}
 \begin{array}{ccc}
 & [F, 1] & \\
 & \downarrow & \\
 [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \\
 & \downarrow [\sigma, 1] & \\
 & [G, 1] & \\
 & \uparrow & \\
 & [\mathcal{A}', [\mathcal{B}, \mathcal{C}]] &
 \end{array}
 \end{array}$$

and

$$\begin{array}{c}
 \begin{array}{ccccc}
 & [F \otimes 1, 1] & & & \\
 & \downarrow & & & \\
 [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}', \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}', [\mathcal{B}, \mathcal{C}]] \\
 & \downarrow [\sigma \otimes 1, 1] & & & \\
 & [G \otimes 1, 1] & & &
 \end{array}
 \end{array}$$

are equal. Composing both 2-cells above on both sides by En , one obtains for the first 2-cell

$$\begin{array}{c}
 \begin{array}{ccc}
 & [F, 1] & \\
 & \downarrow & \\
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A}', [\mathcal{B}, \mathcal{C}]] \\
 & \downarrow [\sigma, 1] & \\
 & [G, 1] & \\
 & \uparrow & \\
 & [\mathcal{A}', \mathcal{B}, \mathcal{C}] &
 \end{array}
 \end{array}$$

since $Rn \circ En = 1$, whereas for the second 2-cell, one obtains

$$\begin{array}{c}
 \begin{array}{ccccccc}
 & [F \otimes 1, 1] & & & & & \\
 & \downarrow & & & & & \\
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{Rn} & [\mathcal{A}', \mathcal{B}, \mathcal{C}] & \xrightarrow{En} & [\mathcal{A}' \otimes \mathcal{B}, \mathcal{C}] \\
 & \downarrow [\sigma \otimes 1, 1] & & & & & \\
 & [G \otimes 1, 1] & & & & &
 \end{array}
 \end{array}$$

which is

$$\begin{array}{c}
 \begin{array}{ccc}
 & [F \otimes 1, 1] & \\
 & \downarrow & \\
 [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \\
 & \downarrow [\sigma \otimes 1, 1] & \\
 & [G \otimes 1, 1] & \\
 & \uparrow & \\
 & [\mathcal{A}' \otimes \mathcal{B}, \mathcal{C}] &
 \end{array}
 \end{array}$$

since the 2-cell $\epsilon * [\sigma \otimes 1, 1] * En$ is an identity. ■

Lemma 16.9 *For any \mathcal{A} and \mathcal{C} , the collection of 1-cells $En_{\mathcal{A}, \mathcal{B}, \mathcal{C}} : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}]$ of SMC defines a 2-natural transformation in the argument \mathcal{B}*

$$\underline{Hom}(\mathcal{A}, -) \circ \underline{Hom}(-, \mathcal{C}) \rightarrow \underline{Hom}(-, \mathcal{C}) \circ \underline{Ten}(\mathcal{A}, -) : SMC^{op} \rightarrow SMC.$$

PROOF: Consider any \mathcal{A} and \mathcal{C} and any 1-cell $F : \mathcal{B}' \rightarrow \mathcal{B}$ in SMC . Let us see that the diagram

$$\begin{array}{ccc} [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] \\ [1, [F, 1]] \downarrow & & \downarrow [1 \otimes F, 1] \\ [\mathcal{A}, [\mathcal{B}', \mathcal{C}]] & \xrightarrow{En} & [\mathcal{A} \otimes \mathcal{B}', \mathcal{C}] \end{array}$$

in SMC commutes. The identity 2-cell corresponding to the equality of 1-cells

$$[\mathcal{A}, [F, \mathcal{C}]] \circ Rn = Rn \circ [\mathcal{A} \otimes F, \mathcal{C}]$$

in SMC given by Lemma 16.4 has for mate

$$En \circ [1, [F, 1]] \rightarrow [1 \otimes F, 1] \circ En : [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] \rightarrow [\mathcal{A} \otimes \mathcal{B}', \mathcal{C}]$$

the 2-cell $\epsilon * ([1 \otimes F, 1] \circ En)$ that is an identity.

The rest of the proof is similar to the end the proof for Lemma 16.8. ■

17 More commuting diagrams

This section gathers technical results involving altogether the internal hom \underline{Hom} , the tensor and the isomorphism D of SMC .

Lemma 17.1 *Given any symmetric monoidal categories $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} , the following diagram in \mathbf{Cat} is commutative*

$$\begin{array}{ccc} SMC(\mathcal{A}, [\mathcal{B}, [\mathcal{C}, \mathcal{D}]]) & \xrightarrow{SMC(-, En)} & SMC(\mathcal{A}, [\mathcal{B} \otimes \mathcal{C}, \mathcal{D}]) \\ D \downarrow & & \downarrow D \\ SMC(\mathcal{B}, [\mathcal{A}, [\mathcal{C}, \mathcal{D}]]) & \xrightarrow{SMC(-, D)} & SMC(\mathcal{B}, [\mathcal{C}, [\mathcal{A}, \mathcal{D}]]) \\ & \nwarrow Rn & \nearrow \\ & SMC(\mathcal{B}, [\mathcal{C}, [\mathcal{A}, \mathcal{D}]]) & \end{array}$$

PROOF: Considering any symmetric monoidal functor $F : \mathcal{A} \rightarrow [\mathcal{B}, [\mathcal{C}, \mathcal{D}]]$, we first show that the composite

$$\mathcal{B} \xrightarrow{F^*} [\mathcal{A}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{D} [\mathcal{C}, [\mathcal{A}, \mathcal{D}]]$$

is the image by Rn of the dual of the composite

$$\mathcal{A} \xrightarrow{F} [\mathcal{B}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{En} [\mathcal{B} \otimes \mathcal{C}, \mathcal{D}].$$

According to Lemma 10.2, $D \circ F^*$ is the composite in SMC

$$\mathcal{B} \xrightarrow{q} [[\mathcal{B}, [\mathcal{C}, \mathcal{D}]], [\mathcal{C}, \mathcal{D}]] \xrightarrow{[F, 1]} [\mathcal{A}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{D} [\mathcal{C}, [\mathcal{A}, \mathcal{D}]]$$

whereas the dual of $En \circ F$ is

$$\mathcal{B} \otimes \mathcal{C} \xrightarrow{q} [[[\mathcal{B} \otimes \mathcal{C}], \mathcal{D}], \mathcal{D}] \xrightarrow{[En, 1]} [[\mathcal{B}, [\mathcal{C}, \mathcal{D}]], \mathcal{D}] \xrightarrow{[F, 1]} [\mathcal{A}, \mathcal{D}].$$

Therefore we need to show the equality of the external legs in the pasting in SMC

$$\begin{array}{ccc}
\mathcal{B} & \xrightarrow{\eta} & [\mathcal{C}, \mathcal{B} \otimes \mathcal{C}] \\
\downarrow q & \swarrow [-, \mathcal{D}] & \downarrow [\mathcal{C}, q] \\
& [[\mathcal{B} \otimes \mathcal{C}, \mathcal{D}], [\mathcal{C}, \mathcal{D}]] & \xleftarrow{D} [\mathcal{C}, [[\mathcal{B} \otimes \mathcal{C}, \mathcal{D}], \mathcal{D}]] \\
& \swarrow [En, 1] & \downarrow [\mathcal{C}, [En, 1]] \\
[[\mathcal{B}, [\mathcal{C}, \mathcal{D}]], [\mathcal{C}, \mathcal{D}]] & \xrightarrow{D} & [\mathcal{C}, [[\mathcal{B}, [\mathcal{C}, \mathcal{D}]], \mathcal{D}]] \\
\downarrow [F, 1] & & \downarrow [\mathcal{C}, [F, 1]] \\
[\mathcal{A}, [\mathcal{C}, \mathcal{D}]] & \xrightarrow{D} & [\mathcal{C}, [\mathcal{A}, \mathcal{D}]]
\end{array}$$

In this pasting all diagrams are commutative. According to Proposition 10.4 the two bottom diagrams commute. The top right triangle commutes according to Lemma 10.3. Eventually to show the commutativity of the top left diagram in the pasting above, consider the dual of

$$\mathcal{B} \xrightarrow{\eta} [\mathcal{C}, \mathcal{B} \otimes \mathcal{C}] \xrightarrow{[-, \mathcal{D}]} [[\mathcal{B} \otimes \mathcal{C}, \mathcal{D}], [\mathcal{C}, \mathcal{D}]] \xrightarrow{[En, 1]} [[\mathcal{B}, [\mathcal{C}, \mathcal{D}]], [\mathcal{C}, \mathcal{D}]].$$

According to Lemma 10.1, this is the arrow

$$[\mathcal{B}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{En} [\mathcal{B} \otimes \mathcal{C}, \mathcal{D}] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, \mathcal{B} \otimes \mathcal{C}], [\mathcal{C}, \mathcal{D}]] \xrightarrow{[\eta, 1]} [\mathcal{B}, [\mathcal{C}, \mathcal{D}]]$$

which is $Rn \circ En$, which is the identity at $[\mathcal{B}, [\mathcal{C}, \mathcal{D}]]$ and the dual of q .

It is rather straightforward to adapt the previous argument and use the 2-naturality of D to show that for any monoidal natural transformation σ between symmetric functors $F \rightarrow G : \mathcal{A} \rightarrow [\mathcal{B}, [\mathcal{C}, \mathcal{D}]]$, the monoidal natural transformation

$$\begin{array}{ccc}
& F^* & \\
& \parallel & \\
\mathcal{B} & \xrightarrow{\sigma^*} & [\mathcal{A}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{D} [\mathcal{C}, [\mathcal{A}, \mathcal{D}]] \\
& \parallel & \\
& G^* &
\end{array}$$

is the image by Rn of the dual of

$$\begin{array}{ccc}
& F & \\
& \parallel & \\
\mathcal{A} & \xrightarrow{\sigma} & [\mathcal{B}, [\mathcal{C}, \mathcal{D}]] \xrightarrow{En} [\mathcal{B} \otimes \mathcal{C}, \mathcal{D}] \\
& \parallel & \\
& G &
\end{array}$$

■

The image by En of the identity at $[\mathcal{B}, \mathcal{C}]$ is an arrow $[\mathcal{B}, \mathcal{C}] \otimes \mathcal{B} \rightarrow \mathcal{C}$ that we denote $Eval$.

Lemma 17.2 *The arrow $[\mathcal{A}, -] : [\mathcal{B}, \mathcal{C}] \rightarrow [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]]$ is equal to the composite*

$$[\mathcal{B}, \mathcal{C}] \xrightarrow{[Eval, \mathcal{C}]} [[\mathcal{A}, \mathcal{B}] \otimes \mathcal{A}, \mathcal{C}] \xrightarrow{Rn} [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]].$$

PROOF: By definition of $Eval$, the following diagram in SMC commutes

$$\begin{array}{ccc} [\mathcal{A}, \mathcal{B}] & \xrightarrow{\eta} & [\mathcal{A}, [\mathcal{A}, \mathcal{B}] \otimes \mathcal{A}] \\ & \searrow 1 & \downarrow [\mathcal{A}, Eval] \\ & & [\mathcal{A}, \mathcal{B}]. \end{array}$$

Consider then the pasting of commutative diagrams in SMC

$$\begin{array}{ccccc} [\mathcal{B}, \mathcal{C}] & \xrightarrow{[Eval, 1]} & [[\mathcal{A}, \mathcal{B}] \otimes \mathcal{A}, \mathcal{C}] & \xrightarrow{Rn} & [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]] \\ \downarrow [\mathcal{A}, -] & & \downarrow [\mathcal{A}, -] & \nearrow [\eta, 1] & \\ [[\mathcal{A}, \mathcal{B}], [\mathcal{A}, \mathcal{C}]] & \xrightarrow{[[1, Eval], 1]} & [[\mathcal{A}, [\mathcal{A}, \mathcal{B}] \otimes \mathcal{A}], [\mathcal{A}, \mathcal{C}]] & & \\ & \searrow 1 & & & \end{array}$$

the top left diagram commuting according to Corollary 9.9.

■

18 The free symmetric monoidal category \mathcal{I} over the terminal category

In this section we consider the free symmetric monoidal category \mathcal{I} over the terminal category $\mathbf{1}$. We show that for any symmetric monoidal category \mathcal{A} , the strict monoidal functor $v : \mathcal{I} \rightarrow [\mathcal{A}, \mathcal{A}]$ sending the generator \star of \mathcal{I} to the identity $\mathcal{A} \rightarrow \mathcal{A}$, has dual v^* which is left adjoint in SMC to $ev_\star : [\mathcal{I}, \mathcal{A}] \rightarrow \mathcal{A}$, the evaluation functor at \star .

The set of objects of \mathcal{I} , which we write \mathcal{T} , is the underlying set of the free algebra over the one point set, with element denoted by \star , for the signature consisting of one constant symbol denoted I and one binary symbol denoted \otimes .

We write \mathcal{K} for the graph with set of vertices \mathcal{T} and set of edges defined by induction according to the following rules.

- For any X, Y, Z in \mathcal{T} , there are one edge $ass_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ and one edge $\overline{ass}_{X,Y,Z} : (X \otimes Y) \otimes Z \rightarrow X \otimes (Y \otimes Z)$.
- For any X in \mathcal{T} , there are one edge $r_X : X \otimes I \rightarrow X$, one edge $\bar{r}_X : X \rightarrow X \otimes I$, one edge $l_X : I \otimes X \rightarrow X$ and one edge $\bar{l}_X : X \rightarrow I \otimes X$.
- For any X, Y in \mathcal{T} , there is one edge $s_{X,Y} : X \otimes Y \rightarrow Y \otimes X$.
- For any X in \mathcal{T} and any edge $p : Y \rightarrow Z$, there are two new edges $X \otimes p : X \otimes Y \rightarrow X \otimes Z$ and $p \otimes X : Y \otimes X \rightarrow Z \otimes X$.
- Edges in \mathcal{K} with different names are different.

We write \mathcal{F} for the free category on \mathcal{K} . For any object X of \mathcal{K} , the graph morphisms $X \otimes -$ and $- \otimes X$ extend uniquely to endofunctors of \mathcal{F} . The underlying category of \mathcal{I} is defined as the quotient category of \mathcal{F} by the congruence generated by a set of relations on the arrows of \mathcal{F} similar to those defining the tensor of Section 12. This time, this set is the smallest set of relations that is closed by the expansions of all relations by $X \otimes -$ and $- \otimes X$ for any object X , and that contains the following.

- Relations expressing that the $\overline{ass}_{X,Y,Z}$ are “inverses” of the $ass_{X,Y,Z}$, the \bar{r}_X are inverses of the r_X , the \bar{l}_X are inverses of the l_X ;
- The “coherence axioms” for symmetric monoidal categories;
- Relations expressing the “naturalities” of the collections of arrows ass , r , l and s ;
- Relations expressing the “bifunctionality of the tensor”, i.e. for any edges $t : X \rightarrow Y$ and $s : Z \rightarrow W$ in \mathcal{K} , the relation $(t \otimes W)(X \otimes s) \sim (Y \otimes s)(t \otimes Z)$.

For any object X of \mathcal{F} , the endofunctors of \mathcal{F} obtained from $X \otimes -$ and $- \otimes X$, induce respectively endofunctors of \mathcal{I} and the collection of such endofunctors defines a functor $\mathcal{I} \times \mathcal{I} \rightarrow \mathcal{I}$. The category \mathcal{I} admits a symmetric monoidal with canonical arrows ass , r , l , s the congruence classes of edges of \mathcal{K} with the same names, with tensor the above bifunctor and with the object I as unit.

For any symmetric monoidal category \mathcal{A} , by the universal property defining \mathcal{I} , there exists a unique strict symmetric monoidal functor $v : \mathcal{I} \rightarrow [\mathcal{A}, \mathcal{A}]$ such that $v(\star)$ is the identity functor $I_{\mathcal{A}}$ in \mathcal{A} with its strict structure. In particular on other objects of \mathcal{I} , the functor v is as follows:

- $v(I) = I_{[\mathcal{A}, \mathcal{A}]}$;
- for any X, Y in \mathcal{T} , $v(X \otimes Y) = v(X) \square v(Y)$.

Note then that since the tensor $F \square G$ of any two strong functors F and G is also strong, one obtains by induction that all the functors $v(X)$ above are strong.

Proposition 18.1 *For any symmetric monoidal category \mathcal{A} , the arrow $v^* : \mathcal{A} \rightarrow [\mathcal{I}, \mathcal{A}]$ in SMC , that is dual of $v : \mathcal{I} \rightarrow [\mathcal{A}, \mathcal{A}]$, has right adjoint in SMC the evaluation at \star functor $ev_{\star} : [\mathcal{I}, \mathcal{A}] \rightarrow \mathcal{A}$. Moreover the composite $ev_{\star} \circ v^*$ is the identity at \mathcal{A} .*

PROOF: We know from Section 11 that the functor “evaluation at \star ” $SMC(\mathcal{I}, \mathcal{A}) \rightarrow \mathcal{A}$ that sends any symmetric monoidal F to its value at the generator \star and any monoidal $\sigma : F \rightarrow G$ to its component in \star , admits a strict monoidal structure $ev_{\star} : [\mathcal{I}, \mathcal{A}] \rightarrow \mathcal{A}$.

The 1-cell $ev_{\star} \circ v^*$ in SMC is the identity at \mathcal{A} since $ev_{\star} \circ v^* = ev_{\star}(v)$ by Lemma 11.9 and one has $ev_{\star}(v) = v(\star) = 1_{[\mathcal{A}, \mathcal{A}]}$.

We show now the existence of a mere adjunction $v^* \dashv ev_{\star} : SMC(\mathcal{I}, \mathcal{A}) \rightarrow \mathcal{A}$.

Given any symmetric monoidal $F : \mathcal{I} \rightarrow \mathcal{A}$, there exists a monoidal natural transformation $\epsilon_F : (v^* \circ ev_{\star})(F) \rightarrow F$. We will freely drop the subscript F for ϵ when there is no ambiguity. It is defined by induction on the structure of the objects of \mathcal{I} according to the following rules:

- $\epsilon_I : v(I)(F(\star)) \rightarrow F(I)$ is $F^0 : I \rightarrow F(I)$;
- $\epsilon_{\star} : v(\star)(F(\star)) \rightarrow F(\star)$ is the identity at $F(\star)$;
- For any objects X, Y of \mathcal{I} , the arrow $\epsilon_{X \otimes Y} : v(X \otimes Y)(F(\star)) \rightarrow F(X \otimes Y)$ is

$$v(X)(F(\star)) \otimes v(Y)(F(\star)) \xrightarrow{\epsilon_X \otimes \epsilon_Y} F(X) \otimes F(Y) \xrightarrow{F_{X,Y}^2} F(X \otimes Y).$$

To check the mere naturality of $\epsilon_F : v^*(ev_{\star}(F)) \rightarrow F : \mathcal{I} \rightarrow \mathcal{A}$ for a given F , it is enough to show that the following diagram in \mathcal{A} commutes

18.2

$$\begin{array}{ccc}
v(X)(F(\star)) & \xrightarrow{v(h)_{F(\star)}} & v(Y)(F(\star)) \\
\epsilon_X \downarrow & & \downarrow \epsilon_Y \\
F(X) & \xrightarrow{F(h)} & F(Y)
\end{array}$$

for all the arrows $h : X \rightarrow Y$ which are equivalence classes of edges of \mathcal{K} .

The commutation of Diagram 18.2 for $h = \text{ass}_{X,Y,Z} : X \otimes (Y \otimes Z) \rightarrow (X \otimes Y) \otimes Z$ amounts to the commutation of the external diagram in the following pasting

$$\begin{array}{ccc}
v(X)(F(\star)) \otimes (v(Y)(F(\star)) \otimes v(Z)(F(\star))) & \xrightarrow{\text{ass}} & (v(X)(F(\star)) \otimes (v(Y)(F(\star)) \otimes v(Z)(F(\star)))) \\
\downarrow \epsilon_X \otimes (\epsilon_Y \otimes \epsilon_Z) & & \downarrow (\epsilon_X \otimes \epsilon_Y) \otimes \epsilon_Z \\
F(X) \otimes (F(Y) \otimes F(Z)) & \xrightarrow{\text{ass}} & (F(X) \otimes F(Y)) \otimes F(Z) \\
\downarrow 1 \otimes F_{Y,Z}^2 & & \downarrow F_{X,Y}^2 \otimes 1 \\
F(X) \otimes F(Y \otimes Z) & & F(X \otimes Y) \otimes F(Z) \\
\downarrow F_{X,Y \otimes Z}^2 & & \downarrow F_{X \otimes Y}^2 \otimes 1 \\
F(X \otimes (Y \otimes Z)) & \xrightarrow{F(\text{ass})} & F((X \otimes Y) \otimes Z).
\end{array}$$

In this pasting, the top diagram commutes according to the naturality of ass and the bottom one also does according to Axiom 2.6 for F .

Similarly one can easily check that Diagram 18.2 commutes for all the instances r , l and s of h according respectively to Axioms 2.7, 2.8 and 2.9 for F , and by naturality of the canonical isomorphisms in \mathcal{A} .

The commutation of Diagram 18.2 for $h = X \otimes p$ for any object X of \mathcal{T} and $p : Y \rightarrow Z$ in \mathcal{K} amounts, since v is strict, to the commutation of the external diagram in the pasting

$$\begin{array}{ccc}
v(X)(F(\star)) \otimes v(Y)(F(\star)) & \xrightarrow{1 \otimes v(p)_{F(\star)}} & v(X)(F(\star)) \otimes v(Z)(F(\star)) \\
\downarrow \epsilon_X \otimes \epsilon_Y & & \downarrow \epsilon_X \otimes \epsilon_Z \\
F(X) \otimes F(Y) & \xrightarrow{1 \otimes F(p)} & F(X) \otimes F(Z) \\
\downarrow F_{X,Y}^2 & & \downarrow F_{X,Z}^2 \\
F(X \otimes Y) & \xrightarrow{F(X \otimes p)} & F(X \otimes Z).
\end{array}$$

In this pasting the bottom diagram commutes by naturality of F^2 in its second argument and the top diagram also does if Diagram 18.2 commutes for $h = p$.

Similarly one has that Diagram 18.2 for $h = p \otimes X$ for any object X of \mathcal{T} and $p : Y \rightarrow Z$ in \mathcal{K} commutes if Diagram 18.2 already commutes for $h = p$.

That for any symmetric monoidal functor F , the natural transformation ϵ_F is monoidal is immediate from its definition. Note also that the transformation ϵ_F is an isomorphism for any strong

F and is the identity for any strict F .

That the collection of arrows ϵ_F for all symmetric monoidal F defines a mere natural transformation between functors

$$\epsilon : v^* \circ ev_\star \rightarrow 1 : SMC(\mathcal{I}, \mathcal{A}) \rightarrow SMC(\mathcal{I}, \mathcal{A})$$

is that for any 2-cell $\sigma : F \rightarrow G : \mathcal{I} \rightarrow \mathcal{A}$ in SMC and any X in \mathcal{I} the following diagram in \mathcal{A} commutes

18.3

$$\begin{array}{ccc} v(X)(F(\star)) & \xrightarrow{(\epsilon_F)_X} & F(X) \\ v(X)(\sigma_\star) \downarrow & & \downarrow \sigma_X \\ v(X)(G(\star)) & \xrightarrow{(\epsilon_G)_X} & G(X). \end{array}$$

We show this by induction on the structure of the objects X of \mathcal{I} .

For $X = I$, the commutation of Diagram 18.3 amounts to the commutation of

$$\begin{array}{ccc} I & \xrightarrow{F^0} & F(I) \\ \parallel & & \downarrow \sigma_I \\ I & \xrightarrow{G^0} & G(I) \end{array}$$

which is Axiom 2.11 for σ .

For $X = \star$, the commutation of 18.3 amounts to the commutation of

$$\begin{array}{ccc} F(\star) & \equiv & F(\star) \\ \sigma_\star \downarrow & & \downarrow \sigma_\star \\ G(\star) & \equiv & G(\star) \end{array}$$

which is trivial.

For $X = Y \otimes Z$, the commutation of 18.3 amounts to the commutation of the external diagram in the pasting

$$\begin{array}{ccccc} v(Y)(F\star) \otimes v(Z)(F\star) & \xrightarrow{(\epsilon_F)_Y \otimes (\epsilon_F)_Z} & F(Y) \otimes F(Z) & \xrightarrow{F^2} & F(Y \otimes Z) \\ v(Y)(\sigma_\star) \otimes v(Z)(\sigma_\star) \downarrow & & \downarrow \sigma_Y \otimes \sigma_Z & & \downarrow \sigma_{Y \otimes Z} \\ v(Y)(G\star) \otimes v(Z)(G\star) & \xrightarrow{(\epsilon_G)_Y \otimes (\epsilon_G)_Z} & G(Y) \otimes G(Z) & \xrightarrow{G^2} & G(Y \otimes Z). \end{array}$$

The right diagram in this pasting commutes according to Axiom 2.10 for σ and the left one commutes if Diagram 18.3 commutes for $X = Y$ and for $X = Z$.

The two triangular equalities amount to the facts that both natural transformations

$$ev_\star * \epsilon : ev_\star \circ v^* \circ ev_\star \rightarrow ev_\star : SMC(\mathcal{I}, \mathcal{A}) \rightarrow \mathcal{A}$$

and

$$\epsilon * v^* : v^* \circ ev_\star \circ v^* \rightarrow v^* : \mathcal{A} \rightarrow SMC(\mathcal{I}, \mathcal{A})$$

are identities.

That $ev_\star * \epsilon$ is the identity is immediate from the definition of ϵ . That $\epsilon * v^*$ is the identity results from the fact for any object a in \mathcal{A} the functor $v^*(a) : \mathcal{I} \rightarrow \mathcal{A}$ is strict by Remark 6.7 and thus $\epsilon_{v^*(a)}$ is an identity.

To check that the previous adjunction lifts to an adjunction $v^* \dashv ev_\star : [\mathcal{I}, \mathcal{A}] \rightarrow \mathcal{A}$ in the 2-category SMC , it remains to check that its counit $\epsilon : v^* \circ ev_\star \rightarrow 1$ is monoidal. This amounts to the following two points.

(1) For any $F, G : \mathcal{I} \rightarrow \mathcal{A}$ in SMC , the diagram in $[\mathcal{I}, \mathcal{A}]$

$$\begin{array}{ccc} (v^* \circ ev_\star(F)) \square (v^* \circ ev_\star(G)) & \xrightarrow{(v^* \circ ev_\star)_{F,G}^2} & v^* \circ ev_\star(F \square G) \\ & \searrow \epsilon_F \square \epsilon_G \quad \swarrow \epsilon_{F \square G} & \\ & F \square G & \end{array}$$

commutes.

(2) The diagram in $[\mathcal{I}, \mathcal{A}]$

$$\begin{array}{ccc} I & \xrightarrow{(v^* \circ ev_\star)^0} & v^* \circ ev_\star(I_{[\mathcal{I}, \mathcal{A}]}) \\ & \searrow & \downarrow \epsilon_{I_{[\mathcal{I}, \mathcal{A}]}} \\ & & I \end{array}$$

commutes.

Point (1) above is equivalent to

(1') For any $F, G : \mathcal{I} \rightarrow \mathcal{A}$ and any X in \mathcal{I} , the diagram in \mathcal{A}

$$\begin{array}{ccc} v(X)(F(\star)) \otimes v(X)(G(\star)) & \xrightarrow{v(X)_{F(\star), G(\star)}^2} & v(X)(F(\star) \otimes G(\star)) \\ & \searrow (\epsilon_F)_X \otimes (\epsilon_G)_X \quad \swarrow (\epsilon_{F \otimes G})_X & \\ & F(X) \otimes G(X) & \end{array}$$

commutes.

Point (2) is equivalent to

(2') For any X in \mathcal{I} , the diagram in \mathcal{A}

$$\begin{array}{ccc} I & \xrightarrow{v(X)^0} & v(X)(I_{\mathcal{A}}) \\ & \searrow & \downarrow (\epsilon_{I_{[\mathcal{I}, \mathcal{A}]}})_X \\ & & I \end{array}$$

commutes.

Both points (1') and (2') can be proved by a straightforward induction on the structure of objects of \mathcal{I} . ■

19 Canonical arrows and canonical diagrams

In this section canonical arrows part of the “lax” symmetric monoidal structure on the 2-category SMC are defined. We make precise the fact that these arrows satisfy the usual coherence properties, are 2-natural and are isomorphisms in a lax sense.

Note that in this section we shall sometimes use for convenience the notation $\mathcal{A}\mathcal{B}$ to denote the tensor $\mathcal{A} \otimes \mathcal{B}$ of any symmetric monoidal categories \mathcal{A} and \mathcal{B} .

Given any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} one has the composite functor

$$\gamma_{\mathcal{A},\mathcal{B},\mathcal{C}}^1 : StrSMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \xrightarrow{\cong} SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \xrightarrow{D} SMC(\mathcal{B}, [\mathcal{A}, \mathcal{C}]) \xrightarrow{\cong} StrSMC(\mathcal{B} \otimes \mathcal{A}, \mathcal{C})$$

By the definition of the tensor in SMC and the 2-naturality of D (Lemma 10.1), the collection of $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C}}^1$ defines a 2-natural transformation between **Cat**-valued functors with domain $SMC^{op} \times SMC^{op} \times StrSMC$. By Yoneda, the 2-natural γ^1 corresponds to a 2-natural transformation, namely $S_{\mathcal{B},\mathcal{A}} : \mathcal{B} \otimes \mathcal{A} \rightarrow \mathcal{A} \otimes \mathcal{B}$ between 2-functors $SMC \times SMC \rightarrow StrSMC$. For any \mathcal{A} and \mathcal{B} , the 1-cell $S_{\mathcal{A},\mathcal{B}} : \mathcal{A} \otimes \mathcal{B} \rightarrow \mathcal{B} \otimes \mathcal{A}$ is the image by En of the 1-cell $\eta^* : \mathcal{A} \rightarrow [\mathcal{B}, \mathcal{B} \otimes \mathcal{A}]$ dual of $\eta : \mathcal{B} \rightarrow [\mathcal{A}, \mathcal{B} \otimes \mathcal{A}]$.

Given any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} and \mathcal{D} , one defines $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}^2$ as the composite functor

$$\begin{array}{c} StrSMC((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}, \mathcal{D}) \\ \downarrow \cong \\ SMC(\mathcal{A} \otimes \mathcal{B}, [\mathcal{C}, \mathcal{D}]) \\ \downarrow Rn \\ SMC(\mathcal{A}, [\mathcal{B}, [\mathcal{C}, \mathcal{D}]]) \\ \downarrow SMC(\mathcal{A}, En) \\ SMC(\mathcal{A}, [\mathcal{B} \otimes \mathcal{C}, \mathcal{D}]) \\ \downarrow \cong \\ StrSMC(\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}), \mathcal{D}) \end{array}$$

and one defines $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}^3$ as the composite functor

$$\begin{array}{c} StrSMC(\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}), \mathcal{D}) \\ \downarrow \cong \\ SMC(\mathcal{A}, [\mathcal{B} \otimes \mathcal{C}, \mathcal{D}]) \\ \downarrow SMC(\mathcal{A}, Rn) \\ SMC(\mathcal{A}, [\mathcal{B}, [\mathcal{C}, \mathcal{D}]]) \\ \downarrow En \\ SMC(\mathcal{A} \otimes \mathcal{B}, [\mathcal{C}, \mathcal{D}]) \\ \downarrow \cong \\ StrSMC((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}, \mathcal{D}). \end{array}$$

According to Lemmas 16.3, 16.4, 16.1, 16.8, 16.9 and 16.7, both collections $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}^2$ and $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}^3$ define 2-natural transformations between 2-functors

$$SMC^{op} \times SMC^{op} \times SMC^{op} \times StrSMC \rightarrow \mathbf{Cat}.$$

Therefore by Yoneda, the collections $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}^2$ and $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{D}}^3$ correspond respectively to 2-natural transformations, namely

$$A_{\mathcal{A},\mathcal{B},\mathcal{C}} : \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \rightarrow (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$$

and

$$A'_{\mathcal{A},\mathcal{B},\mathcal{C}} : (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \rightarrow \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}),$$

both between $StrSMC$ -valued 2-functors with domains $SMC \times SMC \times SMC$.

For any $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in SMC , $A_{\mathcal{A},\mathcal{B},\mathcal{C}}$ is the image of the identity at $(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$ by $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}}^2$. This is to say that it is the image by En of

$$\begin{array}{c} \mathcal{A} \\ \downarrow \eta \\ [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}] \\ \downarrow [\mathcal{B}, \eta] \\ [\mathcal{B}, [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]] \\ \downarrow En \\ [\mathcal{B} \otimes \mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]. \end{array}$$

The 1-cell above is $Rn(A_{\mathcal{A},\mathcal{B},\mathcal{C}})$ and its dual is strict according to Remark 6.7 since for all objects a of \mathcal{A} , the symmetric monoidal functor $Rn(A)(a) : \mathcal{B} \otimes \mathcal{C} \rightarrow (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$ is strict as an image by En .

For any $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in SMC , the 1-cell $A'_{\mathcal{A},\mathcal{B},\mathcal{C}} : (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \rightarrow \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$ is the image of the identity at $\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$ by $\gamma_{\mathcal{A},\mathcal{B},\mathcal{C},\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})}^3$. This is to say that $A'_{\mathcal{A},\mathcal{B},\mathcal{C}} : (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \rightarrow \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$ is the image by $En \circ En$ of the arrow

$$\mathcal{A} \xrightarrow{\eta} [\mathcal{B} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})]].$$

Given any symmetric monoidal categories \mathcal{A} and \mathcal{B} , one defines $\gamma_{\mathcal{A},\mathcal{B}}^4$ as the composite functor

$$StrSMC(\mathcal{A} \otimes \mathcal{I}, \mathcal{B}) \xrightarrow{\cong} SMC(\mathcal{A}, [\mathcal{I}, \mathcal{B}]) \xrightarrow{SMC(\mathcal{A}, ev_*)} SMC(\mathcal{A}, \mathcal{B})$$

and $\gamma_{\mathcal{A},\mathcal{B}}^5$ as the composite functor

$$StrSMC(\mathcal{I} \otimes \mathcal{A}, \mathcal{B}) \xrightarrow{\gamma_{\mathcal{I},\mathcal{A},\mathcal{B}}^1} StrSMC(\mathcal{A} \otimes \mathcal{I}, \mathcal{B}) \xrightarrow{\gamma_{\mathcal{A},\mathcal{B}}^4} SMC(\mathcal{A}, \mathcal{B}).$$

According to Lemma 11.2, both collections $\gamma_{\mathcal{A},\mathcal{B}}^4$ and $\gamma_{\mathcal{A},\mathcal{B}}^5$ are 2-natural transformations between 2-functors $SMC \times StrSMC \rightarrow \mathbf{Cat}$. Therefore by Yoneda $\gamma_{\mathcal{A},\mathcal{B}}^4$ and $\gamma_{\mathcal{A},\mathcal{B}}^5$ correspond respectively to the 2-natural transformations between 2-functors $SMC \rightarrow SMC$

$$R'_\mathcal{A} : \mathcal{A} \xrightarrow{\eta} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] \xrightarrow{ev_*} \mathcal{A} \otimes \mathcal{I}$$

and

$$L'_A : \mathcal{A} \xrightarrow{\eta^*} [\mathcal{I}, \mathcal{I} \otimes \mathcal{A}] \xrightarrow{ev_*} \mathcal{I} \otimes \mathcal{A}$$

that satisfies moreover the following.

Lemma 19.1 *The diagram in SMC*

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{I} & \xrightarrow{S} & \mathcal{I} \otimes \mathcal{A} \\ & \swarrow R' \quad \searrow L' & \\ & \mathcal{A} & \end{array}$$

commutes for any \mathcal{A} .

Given any symmetric monoidal category \mathcal{A} , the symmetric monoidal functor $L_A : \mathcal{I} \otimes \mathcal{A} \rightarrow \mathcal{A}$ is defined as the image by En of the symmetric monoidal functor $v : \mathcal{I} \rightarrow [\mathcal{A}, \mathcal{A}]$ defined in Section 18. The symmetric monoidal functor $R_A : \mathcal{A} \otimes \mathcal{I} \rightarrow \mathcal{A}$ is the image by En of the symmetric monoidal functor $v^* : \mathcal{A} \rightarrow [\mathcal{I}, \mathcal{A}]$, dual of v via the isomorphism 6.2.

When no ambiguity can occur, we shall omit the subscripts $\mathcal{A}, \mathcal{B}, \dots$, for the “canonical” arrows A, A', R, R', L, L' and S .

For any $\mathcal{A}, \mathcal{B}, \mathcal{C}$ in SMC , the composite functor $D_{\mathcal{B}, \mathcal{A}, \mathcal{C}} \circ D_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ is the identity and thus the functor $\gamma_{\mathcal{B}, \mathcal{A}, \mathcal{C}}^1 \circ \gamma_{\mathcal{A}, \mathcal{B}, \mathcal{C}}^1$ is the identity at $StrSMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})$. Therefore by Yoneda, one has the following.

Lemma 19.2 *For any \mathcal{A} and \mathcal{B} , the composite $S_{\mathcal{B}, \mathcal{A}} \circ S_{\mathcal{A}, \mathcal{B}}$ is the identity at $\mathcal{A} \otimes \mathcal{B}$ in SMC .*

Lemma 19.3 *The diagram in SMC*

$$\begin{array}{ccc} [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{[S, 1]} & [\mathcal{B} \otimes \mathcal{A}, \mathcal{C}] \\ Rn \downarrow & & \downarrow Rn \\ [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{D} & [\mathcal{B}, [\mathcal{A}, \mathcal{C}]] \end{array}$$

commutes for any \mathcal{A}, \mathcal{B} and \mathcal{C} .

PROOF: Consider the pasting of diagrams in SMC

$$\begin{array}{ccc} [\mathcal{A} \otimes \mathcal{B}, \mathcal{C}] & \xrightarrow{[S, 1]} & [\mathcal{B} \otimes \mathcal{A}, \mathcal{C}] \\ \downarrow [\mathcal{B}, -] & \searrow [\mathcal{A}, -] & \downarrow [\mathcal{A}, -] \\ [[\mathcal{B}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{B}, \mathcal{C}]] & [[\mathcal{A}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{A}, \mathcal{C}]] \xrightarrow{[[1, S], 1]} [[\mathcal{A}, \mathcal{B} \otimes \mathcal{A}], [\mathcal{A}, \mathcal{C}]] & \\ \downarrow [\eta, 1] & \searrow [\eta^*, 1] & \downarrow [\eta, 1] \\ [\mathcal{A}, [\mathcal{B}, \mathcal{C}]] & \xrightarrow{D} & [\mathcal{B}, [\mathcal{A}, \mathcal{C}]]. \end{array}$$

All the diagrams above are commutative, the top one according to Corollary 9.9 and the left-bottom one according to Lemma 10.8. ■

According to the 2-naturality of S , one can define for any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} the 2-cell $T_{\mathcal{A},\mathcal{B},\mathcal{C}}$ in SMC as either one of the two composites

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \xrightarrow{S} \mathcal{C} \otimes (\mathcal{A} \otimes \mathcal{B}) \xrightarrow{1 \otimes S} \mathcal{C} \otimes (\mathcal{B} \otimes \mathcal{A})$$

or

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \xrightarrow{S \otimes 1} (\mathcal{B} \otimes \mathcal{A}) \otimes \mathcal{C} \xrightarrow{S} \mathcal{C} \otimes (\mathcal{B} \otimes \mathcal{A}).$$

Lemma 19.4 *For any \mathcal{A} , \mathcal{B} and \mathcal{C} , the image by $Rn \circ D \circ Rn$ of the 1-cell $A : \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \rightarrow (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$ in SMC equals to*

$$\mathcal{B} \xrightarrow{\eta^*} [\mathcal{A}, \mathcal{A} \otimes \mathcal{B}] \xrightarrow{[1, \eta]} [\mathcal{A}, [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]] \xrightarrow{D} [\mathcal{C}, [\mathcal{A}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]].$$

PROOF: The image by Rn of $A_{\mathcal{A},\mathcal{B},\mathcal{C}}$ is the 1-cell

$$\mathcal{A} \xrightarrow{\eta} [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}] \xrightarrow{[1, \eta]} [\mathcal{B}, [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]] \xrightarrow{En} [\mathcal{B} \otimes \mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}].$$

According to Lemma 17.1, the image by Rn of its dual is

$$\mathcal{B} \xrightarrow{F^*} [\mathcal{A}, [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]] \xrightarrow{D} [\mathcal{C}, [\mathcal{A}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]]$$

where F is the 1-cell $\mathcal{A} \xrightarrow{\eta} [\mathcal{B}, \mathcal{A} \otimes \mathcal{B}] \xrightarrow{[1, \eta]} [\mathcal{B}, [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]]$. According to Lemma 10.1 this F has dual

$$\mathcal{B} \xrightarrow{\eta^*} [\mathcal{A}, \mathcal{A} \otimes \mathcal{B}] \xrightarrow{[1, \eta]} [\mathcal{A}, [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]]$$

■

According to the previous lemma and Lemmas 10.5 and 10.1, one obtains

Corollary 19.5 *For any \mathcal{A} , \mathcal{B} and \mathcal{C} , the image by $D \circ Rn \circ D \circ Rn$ of the 1-cell $A : \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \rightarrow (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$ in SMC equals to*

$$\mathcal{C} \xrightarrow{\eta^*} [\mathcal{A} \otimes \mathcal{B}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}] \xrightarrow{[\mathcal{A}, -]} [[\mathcal{A}, \mathcal{A} \otimes \mathcal{B}], [\mathcal{A}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]] \xrightarrow{[\eta^*, 1]} [\mathcal{B}, [\mathcal{A}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]].$$

We can now relate the 1-cells A and A' .

Lemma 19.6 *The diagram in SMC*

$$\begin{array}{ccc} (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} & \xrightarrow{A'} & \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \\ T_{\mathcal{A},\mathcal{B},\mathcal{C}} \downarrow & & \uparrow T_{\mathcal{C},\mathcal{B},\mathcal{A}} \\ \mathcal{C} \otimes (\mathcal{B} \otimes \mathcal{A}) & \xrightarrow{A} & (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A} \end{array}$$

commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} .

PROOF: Both legs of the diagram are strict functors. We show that their images by Rn are strict and that their images by $Rn \circ Rn$ are equal.

The 1-cell

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \xrightarrow{S} \mathcal{C} \otimes (\mathcal{A} \otimes \mathcal{B}) \xrightarrow{1 \otimes S} \mathcal{C} \otimes (\mathcal{B} \otimes \mathcal{A}) \xrightarrow{A} (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}$$

has image by Rn

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{\eta^*} [\mathcal{C}, \mathcal{C} \otimes (\mathcal{A} \otimes \mathcal{B})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{C}, \mathcal{C} \otimes (\mathcal{B} \otimes \mathcal{A})] \xrightarrow{[1, A]} [\mathcal{C}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}]$$

which is

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{S} \mathcal{B} \otimes \mathcal{A} \xrightarrow{\eta^*} [\mathcal{C}, \mathcal{C} \otimes (\mathcal{B} \otimes \mathcal{A})] \xrightarrow{[1, A]} [\mathcal{C}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}]$$

or

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{S} \mathcal{B} \otimes \mathcal{A} \xrightarrow{Rn(A)^*} [\mathcal{C}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}].$$

Observe that this 1-cell is strict since both S and $Rn(A)^*$ are. The image by Rn of the 1-cell $(Rn(A))^* \circ S$ above is

$$\mathcal{A} \xrightarrow{\eta^*} [\mathcal{B}, \mathcal{B} \otimes \mathcal{A}] \xrightarrow{[1, Rn(A)^*]} [\mathcal{B}, [\mathcal{C}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}]]$$

which is the dual of $Rn \circ D \circ Rn(A_{\mathcal{C}, \mathcal{B}, \mathcal{A}})$, that is

$$\mathcal{A} \xrightarrow{\eta^*} [\mathcal{C} \otimes \mathcal{B}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, \mathcal{C} \otimes \mathcal{B}], [\mathcal{C}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}]] \xrightarrow{[\eta^*, 1]} [\mathcal{B}, [\mathcal{C}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}]]$$

according to Corollary 19.5.

On the other hand the image by Rn of the 1-cell

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \xrightarrow{A'} \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \xrightarrow{1 \otimes S} \mathcal{A} \otimes (\mathcal{C} \otimes \mathcal{B}) \xrightarrow{S} (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}$$

is the strict 1-cell $[1, S] \circ [1, 1 \otimes S] \circ Rn(A') : \mathcal{A} \otimes \mathcal{B} \rightarrow [\mathcal{C}, (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A}]$ which image by Rn is

$$\mathcal{A} \xrightarrow{\eta} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})]] \xrightarrow{[1, [1, 1 \otimes S]]} [\mathcal{B}, [\mathcal{C}, \mathcal{A}(\mathcal{C}\mathcal{B})]] \xrightarrow{[1, [1, S]]} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]] .$$

This last 1-cell rewrites successively as

1. $\mathcal{A} \xrightarrow{\eta} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{C}\mathcal{B})] \xrightarrow{[1, S]} [\mathcal{B}\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]]$
2. $\mathcal{A} \xrightarrow{\eta} [\mathcal{C}\mathcal{B}, \mathcal{A}(\mathcal{C}\mathcal{B})] \xrightarrow{[S, 1]} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{C}\mathcal{B})] \xrightarrow{[1, S]} [\mathcal{B}\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]]$
3. $\mathcal{A} \xrightarrow{\eta} [\mathcal{C}\mathcal{B}, \mathcal{A}(\mathcal{C}\mathcal{B})] \xrightarrow{[1, S]} [\mathcal{C}\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{[S, 1]} [\mathcal{B}\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]]$
4. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{C}\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{[S, 1]} [\mathcal{B}\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]]$
5. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{C}\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{Rn} [\mathcal{C}, [\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}]] \xrightarrow{D} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]]$
6. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{C}\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{[\mathcal{B}, -]} [[\mathcal{B}, \mathcal{C}\mathcal{B}], [\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}]] \xrightarrow{[\eta, 1]} [\mathcal{C}, [\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}]] \xrightarrow{D} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]]$
7. $\mathcal{A} \xrightarrow{\eta^*} [\mathcal{C}\mathcal{B}, (\mathcal{C}\mathcal{B})\mathcal{A}] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, \mathcal{C}\mathcal{B}], [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]] \xrightarrow{[\eta^*, 1]} [\mathcal{B}, [\mathcal{C}, (\mathcal{C}\mathcal{B})\mathcal{A}]] .$

In the above derivation, arrows 4. and 5. are equal due to Lemma 19.3 and arrows 6. and 7. are equal due to Lemma 10.8. ■

Lemma 19.7 *There exists a 2-cell $A \circ A' \rightarrow 1 : (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \rightarrow (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}$ in SMC, for any \mathcal{A} , \mathcal{B} and \mathcal{C} .*

PROOF: The image by $Rn \circ Rn$ of the composite $A \circ A'$ is

$$\mathcal{A} \xrightarrow{Rn(Rn(A'))} [\mathcal{B}, [\mathcal{C}, \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})]] \xrightarrow{[1, [1, A]]} [\mathcal{B}, [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]]$$

which rewrites successively as

$$\begin{aligned}
\mathcal{A} &\xrightarrow{\eta} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})]] \xrightarrow{[1, [1, \mathcal{A}]}} [\mathcal{B}, [\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}]]. \\
\mathcal{A} &\xrightarrow{\eta} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})] \xrightarrow{[1, \mathcal{A}]} [\mathcal{B}\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}]] \\
\mathcal{A} &\xrightarrow{Rn(\mathcal{A})} [\mathcal{B}\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}]] \\
\mathcal{A} &\xrightarrow{\eta} [\mathcal{B}, \mathcal{A}\mathcal{B}] \xrightarrow{[1, \eta]} [\mathcal{B}, [\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}]] \xrightarrow{En} [\mathcal{B}\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}]] \\
\mathcal{A} &\xrightarrow{\eta} [\mathcal{B}, \mathcal{A}\mathcal{B}] \xrightarrow{[1, \eta]} [\mathcal{B}, [\mathcal{C}, (\mathcal{A}\mathcal{B})\mathcal{C}]]
\end{aligned}$$

Therefore $Rn(Rn(A \circ A'))$ is the image by Rn of $\eta : \mathcal{A} \otimes \mathcal{B} \rightarrow [\mathcal{C}, (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}]$. The 1-cell $Rn(A \circ A')$, that is $[\mathcal{C}, A] \circ Rn(A')$, is strict since the 1-cells A and $Rn(A')$ are both strict. Thus there exists a 2-cell $Rn(A \circ A') \rightarrow \eta$ and since $A \circ A'$ is strict, there exists a 2-cell $A \circ A' \rightarrow 1$. ■

Corollary 19.8 *There exists a 2-cell $A' \circ A \rightarrow 1 : \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \rightarrow \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})$ in SMC, for any \mathcal{A}, \mathcal{B} and \mathcal{C} .*

PROOF:Such a 2-cell is obtained as the pasting

$$\begin{array}{ccccc}
\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) & \xrightarrow{A} & (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} & \xrightarrow{A'} & \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) \\
\downarrow T_{\mathcal{C}, \mathcal{B}, \mathcal{A}}^{-1} & & \downarrow T_{\mathcal{A}, \mathcal{B}, \mathcal{C}} & & \downarrow T_{\mathcal{A}, \mathcal{B}, \mathcal{C}}^{-1} \\
(\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A} & \xrightarrow{A'} & \mathcal{C} \otimes (\mathcal{B} \otimes \mathcal{A}) & \xrightarrow{A} & (\mathcal{C} \otimes \mathcal{B}) \otimes \mathcal{A} \\
& \searrow & \downarrow \text{=} & \nearrow & \\
& & 1 & &
\end{array}$$

where the top diagrams commute according to Lemma 19.6 and the bottom 2-cell comes from Lemma 19.7. ■

Lemma 19.9 *The diagram in SMC*

$$\begin{array}{ccc}
((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \otimes \mathcal{D} & \xrightarrow{A'} & (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{C} \otimes \mathcal{D}) \xrightarrow{A'} \mathcal{A} \otimes (\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})) \\
\downarrow A' \otimes 1 & & \uparrow 1 \otimes A' \\
(\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})) \otimes \mathcal{D} & \xrightarrow{A'} & \mathcal{A} \otimes ((\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D})
\end{array}$$

commutes for any $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} .

PROOF:Both legs of the diagram are strict. We show that their images by Rn are strict, that their images by $Rn \circ Rn$ are strict and that their images by $Rn \circ Rn \circ Rn$ are equal.

The image by Rn of the 1-cell

$$((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \otimes \mathcal{D} \xrightarrow{A'} (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{C} \otimes \mathcal{D}) \xrightarrow{A'} \mathcal{A} \otimes (\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}))$$

is the strict 1-cell

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \xrightarrow{Rn(A')} [\mathcal{D}, (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{C} \otimes \mathcal{D})] \xrightarrow{[1, A']} [\mathcal{D}, \mathcal{A} \otimes (\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}))].$$

This last one has image by Rn

$$\mathcal{AB} \xrightarrow{\eta} [\mathcal{CD}, (\mathcal{AB})(\mathcal{CD})] \xrightarrow{Rn} [\mathcal{C}, [\mathcal{D}, (\mathcal{AB})(\mathcal{CD})]] \xrightarrow{[1, [1, A']]} [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]$$

which rewrites successively as

$$\begin{aligned} \mathcal{AB} &\xrightarrow{\eta} [\mathcal{CD}, (\mathcal{AB})(\mathcal{CD})] \xrightarrow{[1, A']} [\mathcal{CD}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))] \xrightarrow{Rn} [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \\ \mathcal{A} \otimes \mathcal{B} &\xrightarrow{Rn(A')} [\mathcal{C} \otimes \mathcal{D}, \mathcal{A} \otimes (\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}))] \xrightarrow{Rn} [\mathcal{C}, [\mathcal{D}, \mathcal{A} \otimes (\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D}))]]. \end{aligned}$$

This last 1-cell is strict and has image by Rn

$$\mathcal{A} \xrightarrow{Rn(Rn(A'))} [\mathcal{B}, [\mathcal{CD}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{[1, Rn]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]]$$

which rewrites successively as

$$\begin{aligned} 1. \quad \mathcal{A} &\xrightarrow{\eta} [\mathcal{B}(\mathcal{CD}), \mathcal{A}(\mathcal{B}(\mathcal{CD}))] \xrightarrow{[\mathcal{CD}, -]} [[\mathcal{CD}, \mathcal{B}(\mathcal{CD})], [\mathcal{CD}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{[\eta, 1]} [\mathcal{B}, [\mathcal{CD}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{[1, Rn]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \\ 2. \quad \mathcal{A} &\xrightarrow{\eta} [\mathcal{B}(\mathcal{CD}), \mathcal{A}(\mathcal{B}(\mathcal{CD}))] \xrightarrow{[\mathcal{CD}, -]} [[\mathcal{CD}, \mathcal{B}(\mathcal{CD})], [\mathcal{CD}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{[1, Rn]} [[\mathcal{CD}, \mathcal{B}(\mathcal{CD})], [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \dots \\ &\dots \xrightarrow{[\eta, 1]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \\ 3. \quad \mathcal{A} &\xrightarrow{\eta} [\mathcal{B}(\mathcal{CD}), \mathcal{A}(\mathcal{B}(\mathcal{CD}))] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}(\mathcal{CD})], [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, [\mathcal{D}, \mathcal{B}(\mathcal{CD})]], [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \xrightarrow{[Rn, 1]} \dots \\ &\dots [[\mathcal{CD}, \mathcal{B}(\mathcal{CD})], [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \xrightarrow{[\eta, 1]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \end{aligned}$$

the above arrows 2. and 3. being equal according to Lemma 16.2.

On the other hand the 1-cell

$$((\mathcal{AB})\mathcal{C})\mathcal{D} \xrightarrow{A' \otimes 1} (\mathcal{A}(\mathcal{BC}))\mathcal{D} \xrightarrow{A'} \mathcal{A}((\mathcal{BC})\mathcal{D}) \xrightarrow{1 \otimes A'} \mathcal{A}(\mathcal{B}(\mathcal{CD}))$$

has image by Rn

$$(\mathcal{AB})\mathcal{C} \xrightarrow{A'} \mathcal{A}(\mathcal{BC}) \xrightarrow{\eta} [\mathcal{D}, ((\mathcal{A}(\mathcal{BC}))\mathcal{D})] \xrightarrow{[1, A']} [\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})] \xrightarrow{[1, 1 \otimes A']} [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]$$

which is

$$(\mathcal{AB})\mathcal{C} \xrightarrow{A'} \mathcal{A}(\mathcal{BC}) \xrightarrow{Rn(A')} [\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})] \xrightarrow{[1, 1 \otimes A']} [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))].$$

Note that this last 1-cell is strict. It has image by Rn the 1-cell

$$\mathcal{AB} \xrightarrow{Rn(A')} [\mathcal{C}, \mathcal{A}(\mathcal{BC})] \xrightarrow{[1, Rn(A')]} [\mathcal{C}, [\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})]] \xrightarrow{[1, [1, 1 \otimes A']]} [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]$$

which is also strict and has image by Rn

$$\mathcal{A} \xrightarrow{Rn(Rn(A'))} [\mathcal{B}, [\mathcal{C}, \mathcal{A}(\mathcal{BC})]] \xrightarrow{[1, [1, Rn(A')]]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})]]] \xrightarrow{[1, [1, [1, 1 \otimes A']]]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] .$$

This last arrow rewrites successively

$$\begin{aligned} 1. \quad \mathcal{A} &\xrightarrow{\eta} [\mathcal{BC}, \mathcal{A}(\mathcal{BC})] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, \mathcal{A}(\mathcal{BC})]] \xrightarrow{[1, [1, Rn(A')]]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})]]] \xrightarrow{[1, [1, [1, 1 \otimes A']]]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \\ 2. \quad \mathcal{A} &\xrightarrow{\eta} [\mathcal{BC}, \mathcal{A}(\mathcal{BC})] \xrightarrow{[1, Rn(A')]} [\mathcal{BC}, [\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})]] \xrightarrow{[1, [1, 1 \otimes A']]} [\mathcal{BC}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \\ 3. \quad \mathcal{A} &\xrightarrow{\eta} [(\mathcal{BC})\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})] \xrightarrow{Rn} [\mathcal{BC}, [\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})]] \xrightarrow{[1, [1, 1 \otimes A']]} [\mathcal{BC}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \\ 4. \quad \mathcal{A} &\xrightarrow{\eta} [(\mathcal{BC})\mathcal{D}, \mathcal{A}((\mathcal{BC})\mathcal{D})] \xrightarrow{[1, 1 \otimes A']} [(\mathcal{BC})\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))] \xrightarrow{Rn} [\mathcal{BC}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{CD}))]]] \end{aligned}$$

$$\begin{array}{l}
5. \mathcal{A} \xrightarrow{\eta} [\mathcal{B}(\mathcal{C}\mathcal{D}), \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{[A', 1]} [(\mathcal{B}\mathcal{C})\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{Rn} [\mathcal{B}\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \\
6. \mathcal{A} \xrightarrow{\eta} [\mathcal{B}(\mathcal{C}\mathcal{D}), \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{[A', 1]} [(\mathcal{B}\mathcal{C})\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, (\mathcal{B}\mathcal{C})\mathcal{D}], [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{[\eta, 1]} \dots \\
\dots [\mathcal{B}\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \\
7. \mathcal{A} \xrightarrow{\eta} [\mathcal{B}(\mathcal{C}\mathcal{D}), \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}(\mathcal{C}\mathcal{D})], [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{[[1, A'], 1]} [[\mathcal{D}, (\mathcal{B}\mathcal{C})\mathcal{D}], [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{[\eta, 1]} \dots \\
\dots [\mathcal{B}\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \\
8. \mathcal{A} \xrightarrow{\eta} [\mathcal{B}(\mathcal{C}\mathcal{D}), \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}(\mathcal{C}\mathcal{D})], [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{[Rn(A'), 1]} [\mathcal{B}\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{[\mathcal{C}, -]} \dots \\
\dots [[\mathcal{C}, \mathcal{B}\mathcal{C}], [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \xrightarrow{[\eta, 1]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \\
9. \mathcal{A} \xrightarrow{\eta} [\mathcal{B}(\mathcal{C}\mathcal{D}), \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}(\mathcal{C}\mathcal{D})], [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, [\mathcal{D}, \mathcal{B}(\mathcal{C}\mathcal{D})]], [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \dots \\
\dots \xrightarrow{[Rn(Rn(A')), 1]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \\
10. \mathcal{A} \xrightarrow{\eta} [\mathcal{B}(\mathcal{C}\mathcal{D}), \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))] \xrightarrow{[\mathcal{D}, -]} [[\mathcal{D}, \mathcal{B}(\mathcal{C}\mathcal{D})], [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, [\mathcal{D}, \mathcal{B}(\mathcal{C}\mathcal{D})]], [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \dots \\
\dots \xrightarrow{[Rn, 1]} [[\mathcal{C}\mathcal{D}, \mathcal{B}(\mathcal{C}\mathcal{D})], [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]] \xrightarrow{[\eta, 1]} [\mathcal{B}, [\mathcal{C}, [\mathcal{D}, \mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D}))]]].
\end{array}$$

In the above derivation the equalities between arrows 6. and 7. and between arrows 8. and 9. hold due to Corollary 9.9. ■

Corollary 19.10 *The diagram in SMC*

$$\begin{array}{ccccc}
\mathcal{A} \otimes (\mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{D})) & \xrightarrow{A} & (\mathcal{A} \otimes \mathcal{B}) \otimes (\mathcal{C} \otimes \mathcal{D}) & \xrightarrow{A} & ((\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C}) \otimes \mathcal{D} \\
\downarrow 1 \otimes A & & & & \uparrow A \otimes 1 \\
\mathcal{A} \otimes ((\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{D}) & \xrightarrow{A} & & & (\mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C})) \otimes \mathcal{D}
\end{array}$$

commutes for any $\mathcal{A}, \mathcal{B}, \mathcal{C}$ and \mathcal{D} .

PROOF: Consider the two pastings below in SMC where all the diagrams involved commute according to Lemma 19.6 and the naturalities of S and A' .

$$\begin{array}{ccccccc}
\mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D})) & \xrightarrow{1 \otimes A} & \mathcal{A}((\mathcal{B}\mathcal{C})\mathcal{D}) & \xrightarrow{A} & (\mathcal{A}(\mathcal{B}\mathcal{C}))\mathcal{D} & \xrightarrow{A \otimes 1} & ((\mathcal{A}\mathcal{B})\mathcal{C})\mathcal{D} \\
\downarrow 1 \otimes T^{-1} & & \downarrow 1 \otimes T & \searrow T^{-1} & \downarrow T & \searrow T^{-1} \otimes 1 & \downarrow T \otimes 1 \\
\mathcal{A}((\mathcal{D}\mathcal{C})\mathcal{B}) & \xrightarrow{1 \otimes A'} & \mathcal{A}(\mathcal{D}(\mathcal{B}\mathcal{C})) & & (\mathcal{D}(\mathcal{B}\mathcal{C}))\mathcal{A} & \xrightarrow{A'} & \mathcal{D}((\mathcal{B}\mathcal{C})\mathcal{A}) \\
\downarrow S & & \downarrow S & \swarrow (1 \otimes S) \otimes 1 & \downarrow 1 \otimes (S \otimes 1) & \swarrow S & \downarrow S \\
((\mathcal{D}\mathcal{C})\mathcal{B})\mathcal{A} & \xrightarrow{A' \otimes 1} & (\mathcal{D}(\mathcal{C}\mathcal{B}))\mathcal{A} & \xrightarrow{A'} & \mathcal{D}((\mathcal{C}\mathcal{B})\mathcal{A}) & \xrightarrow{1 \otimes A'} & \mathcal{D}(\mathcal{C}(\mathcal{B}\mathcal{A}))
\end{array}$$

$$\begin{array}{ccccccc}
\mathcal{A}(\mathcal{B}(\mathcal{C}\mathcal{D})) & \xrightarrow{A} & (\mathcal{A}\mathcal{B})(\mathcal{C}\mathcal{D}) & \xrightarrow{A} & ((\mathcal{A}\mathcal{B})\mathcal{C})\mathcal{D} \\
\downarrow 1 \otimes T^{-1} & & \downarrow T^{-1} & & \downarrow T & & \downarrow T \otimes 1 \\
\mathcal{A}((\mathcal{D}\mathcal{C})\mathcal{B}) & & ((\mathcal{D}\mathcal{C})\mathcal{B})\mathcal{A} & \xrightarrow{A'} & (\mathcal{D}\mathcal{C})(\mathcal{B}\mathcal{A}) & \xrightarrow{A'} & \mathcal{D}(\mathcal{C}(\mathcal{B}\mathcal{A})) \\
\downarrow S & & \downarrow (S \otimes 1) \otimes 1 & & \downarrow S \otimes 1 & & \downarrow 1 \otimes (S \otimes 1) \\
((\mathcal{D}\mathcal{C})\mathcal{B})\mathcal{A} & \xrightarrow{A'} & (\mathcal{D}\mathcal{C})(\mathcal{B}\mathcal{A}) & \xrightarrow{A'} & \mathcal{D}(\mathcal{C}(\mathcal{B}\mathcal{A}))
\end{array}$$

Lemma 19.11 *The diagram in SMC*

$$\begin{array}{ccccc}
(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} & \xrightarrow{A'} & \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) & \xrightarrow{S} & (\mathcal{B} \otimes \mathcal{C}) \otimes \mathcal{A} \\
S \otimes 1 \downarrow & & & & \downarrow A' \\
(\mathcal{B} \otimes \mathcal{A}) \otimes \mathcal{C} & \xrightarrow{A'} & \mathcal{B} \otimes (\mathcal{A} \otimes \mathcal{C}) & \xleftarrow{1 \otimes S} & \mathcal{B} \otimes (\mathcal{C} \otimes \mathcal{A})
\end{array}$$

commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} .

PROOF: Both legs of this diagram are strict. We show that their images by Rn are strict and that their images by $D \circ Rn \circ Rn$ are equal.

The 1-cell

$$(\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} \xrightarrow{S \otimes 1} (\mathcal{B} \otimes \mathcal{A}) \otimes \mathcal{C} \xrightarrow{A'} \mathcal{B} \otimes (\mathcal{A} \otimes \mathcal{C})$$

has image by Rn

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{S} \mathcal{B} \otimes \mathcal{A} \xrightarrow{\eta} [\mathcal{C}, (\mathcal{B} \otimes \mathcal{A}) \otimes \mathcal{C}] \xrightarrow{[1, A']} [\mathcal{C}, \mathcal{B} \otimes (\mathcal{A} \otimes \mathcal{C})]$$

which is

$$\mathcal{A} \otimes \mathcal{B} \xrightarrow{S} \mathcal{B} \otimes \mathcal{A} \xrightarrow{Rn(A')} [\mathcal{C}, \mathcal{B} \otimes (\mathcal{A} \otimes \mathcal{C})] .$$

This last 1-cell is strict and has image by Rn the 1-cell

$$\mathcal{A} \xrightarrow{\eta^*} [\mathcal{B}, \mathcal{B} \otimes \mathcal{A}] \xrightarrow{[1, Rn(A')]} [\mathcal{B}, [\mathcal{C}, \mathcal{B} \otimes (\mathcal{A} \otimes \mathcal{C})]]$$

which, by Lemma 10.1, has dual

$$\mathcal{B} \xrightarrow{\eta} [\mathcal{A}, \mathcal{B} \otimes \mathcal{A}] \xrightarrow{[1, Rn(A')]} [\mathcal{A}, [\mathcal{C}, \mathcal{B} \otimes (\mathcal{A} \otimes \mathcal{C})]]$$

that is $Rn(Rn(A'))$. On the other hand the 1-cell

$$(\mathcal{A}\mathcal{B})\mathcal{C} \xrightarrow{A'} \mathcal{A}(\mathcal{B}\mathcal{C}) \xrightarrow{S} (\mathcal{B}\mathcal{C})\mathcal{A} \xrightarrow{A'} \mathcal{B}(\mathcal{C}\mathcal{A}) \xrightarrow{1 \otimes S} \mathcal{B}(\mathcal{A}\mathcal{C})$$

has image by Rn

$$\mathcal{A}\mathcal{B} \xrightarrow{Rn(A')} [\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})] \xrightarrow{[1, S]} [\mathcal{C}, (\mathcal{B}\mathcal{C})\mathcal{A}] \xrightarrow{[1, A']} [\mathcal{C}, \mathcal{B}(\mathcal{C}\mathcal{A})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]$$

which is strict. This last 1-cell has image by Rn

$$\mathcal{A} \xrightarrow{\eta} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})]] \xrightarrow{[1, [1, S]]} [\mathcal{B}, [\mathcal{C}, (\mathcal{B}\mathcal{C})\mathcal{A}]] \xrightarrow{[1, [1, A']]} [\mathcal{B}, [\mathcal{C}, \mathcal{B}(\mathcal{C}\mathcal{A})]] \xrightarrow{[1, [1, 1 \otimes S]]} [\mathcal{B}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]$$

which rewrites

$$\begin{array}{l}
\mathcal{A} \xrightarrow{\eta} [\mathcal{B}\mathcal{C}, \mathcal{A}(\mathcal{B}\mathcal{C})] \xrightarrow{[1, S]} [\mathcal{B}\mathcal{C}, (\mathcal{B}\mathcal{C})\mathcal{A}] \xrightarrow{[1, A']} [\mathcal{B}\mathcal{C}, \mathcal{B}(\mathcal{C}\mathcal{A})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{B}\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})] \xrightarrow{Rn} [\mathcal{B}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \\
\mathcal{A} \xrightarrow{\eta^*} [\mathcal{B}\mathcal{C}, (\mathcal{B}\mathcal{C})\mathcal{A}] \xrightarrow{[1, A']} [\mathcal{B}\mathcal{C}, \mathcal{B}(\mathcal{C}\mathcal{A})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{B}\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, \mathcal{B}\mathcal{C}], [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \xrightarrow{[\eta, 1]} [\mathcal{B}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]
\end{array}$$

According to Lemma 10.1 and Lemma 10.5 this last 1-cell has dual

$$\gamma : \mathcal{B} \xrightarrow{\eta} [\mathcal{C}, \mathcal{B}\mathcal{C}] \xrightarrow{[1, F]} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \xrightarrow{D} [\mathcal{A}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]$$

where $F : \mathcal{B} \otimes \mathcal{C} \rightarrow [\mathcal{A}, \mathcal{B} \otimes (\mathcal{A} \otimes \mathcal{C})]$ stands for the 1-cell dual of

$$\mathcal{A} \xrightarrow{\eta^*} [\mathcal{B}\mathcal{C}, (\mathcal{B}\mathcal{C})\mathcal{A}] \xrightarrow{[1, A']} [\mathcal{B}\mathcal{C}, \mathcal{B}(\mathcal{C}\mathcal{A})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{B}\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})].$$

This F is

$$\mathcal{B}\mathcal{C} \xrightarrow{\eta} [\mathcal{A}, (\mathcal{B}\mathcal{C})\mathcal{A}] \xrightarrow{[1, A']} [\mathcal{A}, \mathcal{B}(\mathcal{C}\mathcal{A})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})]$$

and therefore the above 1-cell γ is

$$\mathcal{B} \xrightarrow{\eta} [\mathcal{C}, \mathcal{B}\mathcal{C}] \xrightarrow{[1, \eta]} [\mathcal{C}, [\mathcal{A}, (\mathcal{B}\mathcal{C})\mathcal{A}]] \xrightarrow{[1, [1, A']]} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{C}\mathcal{A})]] \xrightarrow{[1, [1, 1 \otimes S]]} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \xrightarrow{D} [\mathcal{A}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]$$

which rewrites successively as

1. $\mathcal{B} \xrightarrow{Rn(Rn(A'))} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{C}\mathcal{A})]] \xrightarrow{[1, [1, 1 \otimes S]]} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \xrightarrow{D} [\mathcal{A}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]$
2. $\mathcal{B} \xrightarrow{\eta} [\mathcal{C}\mathcal{A}, \mathcal{B}(\mathcal{C}\mathcal{A})] \xrightarrow{Rn} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{C}\mathcal{A})]] \xrightarrow{[1, [1, 1 \otimes S]]} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \xrightarrow{D} [\mathcal{A}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]$
3. $\mathcal{B} \xrightarrow{\eta} [\mathcal{C}\mathcal{A}, \mathcal{B}(\mathcal{C}\mathcal{A})] \xrightarrow{[1, 1 \otimes S]} [\mathcal{C}\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})] \xrightarrow{Rn} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \xrightarrow{D} [\mathcal{A}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]$
4. $\mathcal{B} \xrightarrow{\eta} [\mathcal{A}\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})] \xrightarrow{[S, 1]} [\mathcal{C}\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})] \xrightarrow{Rn} [\mathcal{C}, [\mathcal{A}, \mathcal{B}(\mathcal{A}\mathcal{C})]] \xrightarrow{D} [\mathcal{A}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]]$
5. $\mathcal{B} \xrightarrow{\eta} [\mathcal{A}\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})] \xrightarrow{Rn} [\mathcal{A}, [\mathcal{C}, \mathcal{B}(\mathcal{A}\mathcal{C})]].$

In the above derivation arrows 4. and 5. are equal due to Lemma 19.3. ■

Corollary 19.12 *The diagram in SMC*

$$\begin{array}{ccccc} \mathcal{A} \otimes (\mathcal{B} \otimes \mathcal{C}) & \xrightarrow{A} & (\mathcal{A} \otimes \mathcal{B}) \otimes \mathcal{C} & \xrightarrow{S} & \mathcal{C} \otimes (\mathcal{A} \otimes \mathcal{B}) \\ \downarrow 1 \otimes S & & & & \downarrow A \\ \mathcal{A} \otimes (\mathcal{C} \otimes \mathcal{B}) & \xrightarrow{A} & (\mathcal{A} \otimes \mathcal{C}) \otimes \mathcal{B} & \xleftarrow{S \otimes 1} & (\mathcal{C} \otimes \mathcal{A}) \otimes \mathcal{B} \end{array}$$

commutes for any \mathcal{A} , \mathcal{B} and \mathcal{C} .

PROOF: Consider the pasting of commutative diagrams in SMC

$$\begin{array}{ccccc} \mathcal{A}(\mathcal{B}\mathcal{C}) & \xrightarrow{A} & (\mathcal{A}\mathcal{B})\mathcal{C} & \xrightarrow{S} & \mathcal{C}(\mathcal{A}\mathcal{B}) \\ \downarrow T^{-1} & \searrow & \downarrow T & & \swarrow T^{-1} \\ & (\mathcal{C}\mathcal{B})\mathcal{A} & \xrightarrow{A'} & \mathcal{C}(\mathcal{B}\mathcal{A}) & \xrightarrow{S} & (\mathcal{B}\mathcal{A})\mathcal{C} \\ \downarrow 1 \otimes S & \downarrow S \otimes 1 & & \downarrow A' & & \downarrow A \\ & (\mathcal{B}\mathcal{C})\mathcal{A} & \xrightarrow{A'} & \mathcal{B}(\mathcal{C}\mathcal{A}) & \xleftarrow{1 \otimes S} & \mathcal{B}(\mathcal{A}\mathcal{C}) \\ \downarrow T^{-1} & \uparrow T & & \downarrow T & & \downarrow A \\ \mathcal{A}(\mathcal{C}\mathcal{B}) & \xrightarrow{A} & (\mathcal{A}\mathcal{C})\mathcal{B} & \xleftarrow{S \otimes 1} & (\mathcal{C}\mathcal{A})\mathcal{B}. \end{array}$$

Lemma 19.13 *The diagram in SMC*

$$\begin{array}{ccc} \mathcal{A} \otimes \mathcal{I} & \xrightarrow{S} & \mathcal{I} \otimes \mathcal{A} \\ & \searrow R \quad \swarrow L & \\ & \mathcal{A} & \end{array}$$

commutes for any \mathcal{A} .

PROOF: By definition of L and S , the functor $L \circ S$ is strict and the two diagrams involved in the pasting in *SMC* below commute

$$\begin{array}{ccc} \mathcal{A} & \xrightarrow{\eta} & [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] \\ \parallel & & \downarrow [\mathcal{I}, S] \\ \mathcal{A} & \xrightarrow{\eta^*} & [\mathcal{I}, \mathcal{I} \otimes \mathcal{A}] \\ & \searrow v^* & \downarrow [\mathcal{I}, L] \\ & & [\mathcal{I}, \mathcal{A}]. \end{array}$$

This shows that $L \circ S$ is the image by En of v^* . By definition, R is also that image. ■

Lemma 19.14 *The diagram in SMC*

$$\begin{array}{ccc} (\mathcal{A} \otimes \mathcal{I}) \otimes \mathcal{C} & \xrightarrow{A'} & \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C}) \\ & \searrow R' \otimes 1 \quad \swarrow 1 \otimes L' & \\ & \mathcal{A} \otimes \mathcal{C} & \end{array}$$

commutes for all \mathcal{A} and \mathcal{C} .

PROOF: Both legs of the diagram are strict, we show that their images by Rn are equal.

The image by Rn of $A' \circ (R' \otimes 1)$ is

1. $\mathcal{A} \xrightarrow{R'} \mathcal{A} \otimes \mathcal{I} \xrightarrow{\eta} [\mathcal{C}, (\mathcal{A} \otimes \mathcal{I}) \otimes \mathcal{C}] \xrightarrow{[1, A']} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
2. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] \xrightarrow{ev_*} \mathcal{A} \otimes \mathcal{I} \xrightarrow{Rn(A')} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
3. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] \xrightarrow{[1, Rn(A')]} [\mathcal{I}, [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{ev_*} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
4. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{Rn} [\mathcal{I}, [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{ev_*} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
5. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[\mathcal{C}, -]} [[\mathcal{C}, \mathcal{I} \otimes \mathcal{C}], [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{[\eta, 1]} [\mathcal{I}, [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{D} \dots$
 $\dots [\mathcal{C}, [\mathcal{I}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{[1, ev_*]} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
6. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[\mathcal{I}, -]} [[\mathcal{I}, \mathcal{I} \otimes \mathcal{C}], [\mathcal{I}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{[\eta^*, 1]} [\mathcal{C}, [\mathcal{I}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{[1, ev_*]} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
7. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[\mathcal{I}, -]} [[\mathcal{I}, \mathcal{I} \otimes \mathcal{C}], [\mathcal{I}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]] \xrightarrow{[1, ev_*]} [[\mathcal{I}, \mathcal{I} \otimes \mathcal{C}], \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[\eta^*, 1]} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
8. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[ev_*, 1]} [[\mathcal{I}, \mathcal{I} \otimes \mathcal{C}], \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[[1, S], 1]} [[\mathcal{I}, \mathcal{C} \otimes \mathcal{I}], \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[\eta, 1]} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
9. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[S, 1]} [\mathcal{C} \otimes \mathcal{I}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[ev_*, 1]} [[\mathcal{I}, \mathcal{C} \otimes \mathcal{I}], \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[\eta, 1]} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
10. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[S, 1]} [\mathcal{C} \otimes \mathcal{I}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[R', 1]} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$
11. $\mathcal{A} \xrightarrow{\eta} [\mathcal{I} \otimes \mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})] \xrightarrow{[L', 1]} [\mathcal{C}, \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C})]$

and this last arrow is the image of L' by Rn .

In the above derivation:

- arrows 2 and 3 are equal by Corollary 11.2,
- arrows 4 and 5 are equal by Lemma 11.9,
- arrows 5 and 6 are equal by Lemma 10.8,
- arrows 7 and 8 are equal because $Rn(S) = \eta^*$ and by Corollary 11.3,
- arrows 8 and 9 are equal by Corollary 11.2.

Corollary 19.15 *The diagram in SMC*

$$\begin{array}{ccc}
 \mathcal{A} \otimes (\mathcal{I} \otimes \mathcal{C}) & \xrightarrow{A} & (\mathcal{A} \otimes \mathcal{I}) \otimes \mathcal{C} \\
 & \nwarrow 1 \otimes L' \quad \nearrow R' \otimes 1 & \\
 & \mathcal{A} \otimes \mathcal{C} &
 \end{array}$$

commutes for all \mathcal{A} and \mathcal{C} .

PROOF: Consider the pasting in SMC where all diagrams commute.

$$\begin{array}{c}
 \mathcal{A}(\mathcal{I}\mathcal{C}) \xrightarrow{A} (\mathcal{A}\mathcal{I})\mathcal{C} \\
 \downarrow T^{-1} \quad \quad \quad \downarrow T \\
 \begin{array}{ccccc}
 \mathcal{A}(\mathcal{I}\mathcal{C}) & \xrightarrow{A'} & \mathcal{C}(\mathcal{I}\mathcal{A}) & \xrightarrow{S \otimes 1} & (\mathcal{A}\mathcal{I})\mathcal{C} \\
 \downarrow 1 \otimes S & \downarrow S & \downarrow S & \downarrow S & \downarrow S \\
 \mathcal{A}(\mathcal{C}\mathcal{I}) & \xrightarrow{A'} & \mathcal{C}(\mathcal{I}\mathcal{A}) & \xrightarrow{S \otimes 1} & (\mathcal{A}\mathcal{I})\mathcal{C} \\
 \downarrow 1 \otimes L' & \downarrow R' \otimes 1 & \downarrow 1 \otimes L' & \downarrow S & \downarrow S \\
 \mathcal{A}(\mathcal{C}\mathcal{I}) & \xrightarrow{A'} & \mathcal{C}(\mathcal{I}\mathcal{A}) & \xrightarrow{S \otimes 1} & (\mathcal{A}\mathcal{I})\mathcal{C} \\
 \downarrow 1 \otimes R' & \downarrow L' \otimes 1 & \downarrow L' \otimes 1 & \downarrow L' \otimes 1 & \downarrow L' \otimes 1 \\
 \mathcal{A}\mathcal{C} & \xrightarrow{A'} & \mathcal{C}\mathcal{A} & \xrightarrow{L' \otimes 1} & (\mathcal{A}\mathcal{I})\mathcal{C} \\
 \downarrow 1 \otimes L' & \downarrow 1 \otimes R' & \downarrow 1 \otimes R' & \downarrow 1 \otimes R' & \downarrow 1 \otimes R' \\
 \mathcal{A} & \xrightarrow{A'} & \mathcal{C} & \xrightarrow{L' \otimes 1} & (\mathcal{A}\mathcal{I})\mathcal{C}
 \end{array}
 \end{array}$$

For any $F : \mathcal{A} \rightarrow \mathcal{B}$ the following diagram in SMC

$$\begin{array}{ccc}
 [\mathcal{I}, \mathcal{A}] & \xrightarrow{[1, F]} & [\mathcal{I}, \mathcal{B}] \\
 \downarrow ev_* & & \downarrow ev_* \\
 \mathcal{A} & \xrightarrow{F} & \mathcal{B}
 \end{array}$$

commutes according to Corollary 11.2, so that by considering the mate of this identity 2-cell one obtains a 2-cell, namely δ_F as follows

19.16

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{F} & \mathcal{B} \\
 \downarrow v^* & \searrow \delta_F & \downarrow v^* \\
 [\mathcal{I}, \mathcal{A}] & \xrightarrow{[1, F]} & [\mathcal{I}, \mathcal{B}]
 \end{array}$$

Lemma 19.17 *In SMC , one has the following for any \mathcal{A} .*

- The composite $\mathcal{A} \xrightarrow{R'} \mathcal{A} \otimes \mathcal{I} \xrightarrow{R} \mathcal{A}$ is the identity at \mathcal{A} .
- There exists a 1-cell $F : \mathcal{A} \otimes \mathcal{I} \rightarrow \mathcal{A} \otimes \mathcal{I}$ with two 2-cells $F \rightarrow 1 : \mathcal{A} \otimes \mathcal{I} \rightarrow \mathcal{A} \otimes \mathcal{I}$ and $F \rightarrow R' \circ R$.

PROOF: To check that the composite $R \circ R'$ is the identity at \mathcal{A} in SMC , consider the pasting in SMC of commutative diagrams

$$\begin{array}{ccccc}
 & & v^* & & \\
 & \swarrow & & \searrow & \\
 \mathcal{A} & \xrightarrow{\eta} & [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] & \xrightarrow{[1, R]} & [\mathcal{I}, \mathcal{A}] \\
 & \searrow R' & \downarrow ev_* & & \downarrow ev_* \\
 & & \mathcal{A} \otimes \mathcal{I} & \xrightarrow{R} & \mathcal{A}
 \end{array}$$

the bottom right diagram here commuting according to Corollary 11.2. One concludes since $ev_* \circ v^* = 1$ as shown in Proposition 18.1.

We prove now the existence of a 1-cell $F : \mathcal{A} \otimes \mathcal{I} \rightarrow \mathcal{A} \otimes \mathcal{I}$ in SMC with two 2-cells $F \rightarrow 1$ and $F \rightarrow R' \circ R$. For this, we exhibit a 1-cell $G : \mathcal{A} \rightarrow [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}]$ with two 2-cells one from G to η and the other one from G to the composite

$$\mathcal{A} \xrightarrow{v^*} [\mathcal{I}, \mathcal{A}] \xrightarrow{[1, R']} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}]$$

which is actually the image by Rn of $R' \circ R$. To see that this is sufficient, suppose that such 2-cells exist. Then let $F : \mathcal{A} \otimes \mathcal{I} \rightarrow \mathcal{A} \otimes \mathcal{I}$ be the image by En of G . Since the image by En of η is the identity at $\mathcal{A} \otimes \mathcal{I}$ one obtains a 2-cell $F \rightarrow 1_{\mathcal{A} \otimes \mathcal{I}}$. The 2-cell $G \rightarrow Rn(R' \circ R)$ corresponds to a 2-cell $F \rightarrow R' \circ R$ via the adjunction $En \dashv Rn$.

The 1-cell G in question is

$$\mathcal{A} \xrightarrow{R'} \mathcal{A} \otimes \mathcal{I} \xrightarrow{v^*} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}]$$

and the 2-cells are

$$\begin{array}{ccccc}
 & & 1 & & \\
 & \swarrow & \uparrow \epsilon & \searrow & \\
 \mathcal{A} & \xrightarrow{\eta} & [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] & \xrightarrow{ev_*} & \mathcal{A} \otimes \mathcal{I} \xrightarrow{v^*} [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}] \\
 & \searrow & \downarrow & & \\
 & & \mathcal{A} \otimes \mathcal{I} & & \\
 & \swarrow & & \searrow & \\
 & & = & & \\
 & \swarrow & & \searrow & \\
 & & \mathcal{A} & & \\
 & \swarrow & & \searrow & \\
 & & R' & &
 \end{array}$$

where ϵ is the co-unit of the adjunction of Proposition 18.1 and the $\delta_{R'}$

$$\begin{array}{ccc}
 \mathcal{A} & \xrightarrow{R'} & \mathcal{A} \otimes \mathcal{I} \\
 v^* \downarrow & \swarrow & \downarrow v^* \\
 [\mathcal{I}, \mathcal{A}] & \xrightarrow{[1, R']} & [\mathcal{I}, \mathcal{A} \otimes \mathcal{I}]
 \end{array}$$

defined in 19.16. ■

20 A symmetric monoidal closed structure on $SMC_{/\sim}$

This section contains a proof of the following result: the category $SMC_{/\sim}$, quotient of SMC by the congruence generated by its 2-cells admits a symmetric monoidal closed structure.

Given any small category \mathcal{C} , consider the equivalence on its set of objects generated by the relation consisting of the pairs (x, y) such that there exists an arrow $x \rightarrow y$ in \mathcal{C} . Its classes are the so-called *connected components* of \mathcal{C} and we write $[x]$ for the connected component of the object x . The set of connected components of \mathcal{C} is denoted $\pi(\mathcal{C})$. Given any functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, one obtains a map $\pi(F) : \pi(\mathcal{C}) \rightarrow \pi(\mathcal{C}')$ sending any connected component $[x]$ to $[F(x)]$. Note then that:

- For any functors $F : \mathcal{A} \rightarrow \mathcal{B}$ and $G : \mathcal{B} \rightarrow \mathcal{C}$ one has $\pi(G \circ F) = \pi(G) \circ \pi(F)$;
- For any category \mathcal{A} , the map $\pi(1_{\mathcal{A}})$ for the identity functor $1_{\mathcal{A}}$ at \mathcal{A} is the identity map at $\pi(\mathcal{A})$;
- For any natural transformation $\sigma : F \rightarrow G : \mathcal{A} \rightarrow \mathcal{B}$, one has $\pi(F) = \pi(G) : \pi(\mathcal{A}) \rightarrow \pi(\mathcal{B})$.

For convenience we shall consider further mere categories as locally discrete 2-categories and mere functors as 2-functors. With this convention, according to the remark above, the assignments π above define a 2-functor $\pi : \mathbf{Cat} \rightarrow \mathbf{Set}$.

Given a 2-category A , consider the equivalence \sim on its 1-cells generated by the relation consisting of the pairs (f, g) with same domains and codomains and such that there exists a 2-cell $f \rightarrow g$. This equivalence is compatible with the composition of 1-cells. We write $A_{/\sim}$ for the category with the same objects as A , with arrows $x \rightarrow y$, the equivalence classes $f \sim$ by \sim of 1-cells $f : x \rightarrow y$ in A , and with identities and composition induced by those of A :

- For any object x of A , the identity at x in $A_{/\sim}$ is the \sim -class of the identity at x in A ;

- For any arrows $\xrightarrow{f} \xrightarrow{g}$ in A , the composite $g \sim \circ f \sim$ in $A_{/\sim}$ is the \sim -class of $g \circ f$.

One has a 2-functor $p_A : A \rightarrow A_{/\sim}$, that is the identity on the set of objects, sends any 1-cell $f : x \rightarrow y$ to its equivalence class $f \sim$ by \sim and any 2-cell to an identity. For any 2-functor $F : A \rightarrow B$, there exists a unique functor $F_{/\sim} : A_{/\sim} \rightarrow B_{/\sim}$ that renders commutative the diagram of 2-functors

$$\begin{array}{ccc} A & \xrightarrow{F} & B \\ p_A \downarrow & & \downarrow p_B \\ A_{/\sim} & \xrightarrow{F_{/\sim}} & B_{/\sim}. \end{array}$$

Remark 20.1 For any 2-category A , the following assertions hold.

- (i) For any objects x, y of A , there is an isomorphism between sets

$$A_{/\sim}(x, y) \cong \pi(A(x, y))$$

(sending $f \sim$ to $[f]$, for any 1-cell $f : x \rightarrow y$).

- (ii) For any 1-cell $f : x' \rightarrow x$ in A , the following diagram of maps commutes

$$\begin{array}{ccc} \pi(A(x, y)) & \xrightarrow{\pi(A(f, y))} & \pi(A(x', y)) \\ \cong \downarrow & & \downarrow \cong \\ A_{/\sim}(x, y) & \xrightarrow{A_{/\sim}(f \sim, y)} & A_{/\sim}(x', y). \end{array}$$

(iii) For any 1-cell $g : y \rightarrow y'$ in A , the following diagram of maps commutes

$$\begin{array}{ccc} \pi(A(x, y)) & \xrightarrow{\pi(A(x, g))} & \pi(A(x, y')) \\ \cong \downarrow & & \downarrow \cong \\ A_{/\sim}(x, y) & \xrightarrow{A_{/\sim}(x, g)} & A_{/\sim}(x, y'). \end{array}$$

Note that for any two arbitrary 2-categories A and B , the category $(A_{/\sim}) \times (B_{/\sim})$ is isomorphic to $(A \times B)_{/\sim}$. Therefore from any 2-functor $H : A \times B \rightarrow C$ one obtains a functor

$$\tilde{H} : (A_{/\sim}) \times (B_{/\sim}) \xrightarrow{\cong} (A \times B)_{/\sim} \xrightarrow{H_{/\sim}} C_{/\sim}.$$

which is more concretely the following:

- For any objects x of A and y of B , $\tilde{H}(x, y)$ is $H(x, y)$;
- For any 1-cells $f : x \rightarrow x'$ in A and $g : y \rightarrow y'$ in B , $\tilde{H}(f^\sim, g^\sim)$ is $H(f, g)^\sim$.

In particular for the 2-functor $\underline{Ten} : SMC \times SMC \rightarrow SMC$ defined in Section 15, one obtains the functor $\underline{\tilde{Ten}} : SMC_{/\sim} \times SMC_{/\sim} \rightarrow SMC_{/\sim}$ which we denote by the 2-ary symbol $@$.

We can now formulate our result.

Theorem 20.2 *The category $SMC_{/\sim}$ admits the symmetric monoidal closed structure*

$$(SMC_{/\sim}, @, \mathcal{I}, A^\sim, R^\sim, L^\sim, S^\sim).$$

For any symmetric monoidal category \mathcal{B} , the right adjoint to $-@ \mathcal{B} : SMC_{/\sim} \rightarrow SMC_{/\sim}$ sends any symmetric monoidal category \mathcal{C} to $[\mathcal{B}, \mathcal{C}]$ and for any symmetric monoidal functor $F : \mathcal{C} \rightarrow \mathcal{C}'$, sends F^\sim to $[\mathcal{B}, F]^\sim : [\mathcal{B}, \mathcal{C}] \rightarrow [\mathcal{B}, \mathcal{C}']$.

PROOF: According to the 2-functoriality of p_{SMC} , one obtains the following results. The arrows A^\sim are isomorphisms with inverses the A'^\sim according to Lemma 19.7 and Corollary 19.8. The R^\sim are isomorphisms with inverses the R'^\sim according to Lemma 19.17. The collection of arrows $(A_{\mathcal{A}, \mathcal{B}, \mathcal{C}})^\sim$ (respectively $(A'_{\mathcal{A}, \mathcal{B}, \mathcal{C}})^\sim$) is natural in \mathcal{A} , \mathcal{B} and \mathcal{C} since the collection of $A_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$ (respectively $A'_{\mathcal{A}, \mathcal{B}, \mathcal{C}}$) is natural in \mathcal{A} , \mathcal{B} and \mathcal{C} . The collections $(R'_{\mathcal{A}})^\sim$ and $(R_{\mathcal{A}})^\sim$ are natural in \mathcal{A} according to the naturality of the collection of $R'_{\mathcal{A}}$. The collection $(L'_{\mathcal{A}})^\sim$ is natural in \mathcal{A} according to the naturality of the collection of $L'_{\mathcal{A}}$. The arrows L^\sim have inverses the L'^\sim since the R^\sim and R'^\sim are inverses and according to Lemmas 19.1, 19.13 and 19.2. Therefore the collection $(L_{\mathcal{A}})^\sim$ is also natural in \mathcal{A} . The collection of $(S_{\mathcal{A}, \mathcal{B}})^\sim$ is natural in \mathcal{A} and \mathcal{B} since the collection $S_{\mathcal{A}, \mathcal{B}}$ is. Then the coherence axioms 2.1, 2.2, 2.3, 2.4 and 2.5 hold for $ass = A^\sim$, $r = R^\sim$, $l = L^\sim$ and $s = S^\sim$ according respectively to the points 19.10, 19.14, 19.2, 19.12 and 19.13.

Let us consider any symmetric monoidal categories \mathcal{A} , \mathcal{B} and \mathcal{C} . According to Proposition 14.10, one has 2-cells in **Cat**

$$1 \rightarrow Rn \circ En : SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \rightarrow SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}])$$

and

$$En \circ Rn \rightarrow 1 : SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) \rightarrow SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}).$$

Their images by $\pi : \mathbf{Cat} \rightarrow \mathbf{Set}$ being identities, one obtains according to Remark 20.1(i) the isomorphism in **Set**

20.3

$$SMC_{/\sim}(\mathcal{A} @ \mathcal{B}, \mathcal{C}) \cong \pi(SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C})) \xrightarrow{\pi(Rn)} \pi(SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}])) \cong SMC_{/\sim}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]).$$

This collection of isomorphisms is actually natural in \mathcal{A} . To see this, consider any symmetric monoidal functor $F : \mathcal{A}' \rightarrow \mathcal{A}$, and any symmetric monoidal categories \mathcal{B} and \mathcal{C} . According to Lemma 16.3, the following diagram in **Cat** is commutative

$$\begin{array}{ccc} SMC(\mathcal{A} \otimes \mathcal{B}, \mathcal{C}) & \xrightarrow{Rn} & SMC(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \\ SMC(F \otimes 1, 1) \downarrow & & \downarrow SMC(F, 1) \\ SMC(\mathcal{A}' \otimes \mathcal{B}, \mathcal{C}) & \xrightarrow{Rn} & SMC(\mathcal{A}', [\mathcal{B}, \mathcal{C}]). \end{array}$$

By definition of \otimes and according to Remark 20.1 (i) and (ii), applying π to the above yields the diagram of maps

$$\begin{array}{ccc} SMC_{/\sim}(\mathcal{A} @ \mathcal{B}, \mathcal{C}) & \xrightarrow{\cong} & SMC_{/\sim}(\mathcal{A}, [\mathcal{B}, \mathcal{C}]) \\ SMC_{/\sim}(F^{\sim} @ 1, 1) \downarrow & & \downarrow SMC_{/\sim}(F^{\sim}, 1) \\ SMC_{/\sim}(\mathcal{A}' @ \mathcal{B}, \mathcal{C}) & \xrightarrow{\cong} & SMC_{/\sim}(\mathcal{A}', [\mathcal{B}, \mathcal{C}]). \end{array}$$

For any symmetric monoidal category \mathcal{B} there exists a unique functor $SMC_{/\sim} \rightarrow SMC_{/\sim}$ taking values $[\mathcal{B}, \mathcal{C}]$ for any symmetric monoidal category \mathcal{C} and that renders the collection of isomorphisms 20.3 also natural in \mathcal{C} . For any arrow $F : \mathcal{C} \rightarrow \mathcal{C}'$, this functor sends the class F^{\sim} to the image by $\pi(Rn)$ of

$$[\mathcal{B}, \mathcal{C}] \otimes \mathcal{B} \xrightarrow{Eval^{\sim}} \mathcal{C} \xrightarrow{F^{\sim}} \mathcal{C}'$$

and this image is $[\mathcal{B}, F]^{\sim}$ according to Lemma 17.2. ■

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